ex. 2

(c) $f : \mathbb{N} \to \mathbb{N}$, f incr., $Rng(f) = X$ is recursive, then $f \in Rec$.

<u>Proof</u> Let $ge(x,y) = \begin{cases} 1 & x \geq y \\ 0 & \text{otherwise} \end{cases}$.

Obviously $ge \in Rec$. Let $g(n,x) = \chi_X(x) \cdot ge(x,n)$,

i.e. $g(n,x) = 1 \iff x \in X \wedge x \geq n$.

Let $h(n) = \min\{x : g(n,x) = 1\}$, i.e.

$h(n)$ is equal to the smallest $x \in X$ s.t. $x \geq n$.

Let $\begin{cases} i(0) = 0 \\ i(n+1) = \min\{y : \underbrace{h(y) > h(i(n))}\} \end{cases}$

i.e. $i(n+1)$ is a first position in h that is larger than $h(i(n))$.

composition of $>$ and h with args y and $i(n)$

For example: $X = \{1, 3, 7, 8, 10, 11, \dots\}$

n:	0	1	2	3	4	5	6	7	8	9	10	11	...
$h(n)$:	1	1	3	3	7	7	7	7	8	10	10	11	...

$i(n)$: 0 2 4 8 9 11 ...

Then $\bar{f}(n) = h(i(n)) \in Rec$

- $\bar{f}$ is increasing
- $Rng(\bar{f}) = X$

$\Big\} \Rightarrow \bar{f} = f$.

Obvious

$\subseteq$ is obvious. $\supseteq$:

Induction on n where $X = \{x_0, x_1, \dots\}$: $x_0 < x_1 < \dots$

$h(i(n)) = x_n$:

- $h(i(0)) = h(0) = x_0$ ✓
- $h(i(n+1)) = h(\min\{y : h(y) > \underbrace{h(i(n))}_{x_n \text{ by ind. hyp.}}\})$

$= h(\min\{y : h(y) > x_n\})$

$= h(\min\{y :$ smallest $x \in X$ s.t. $x \geq y$ is also greater than $x_n\})$

$= h(x_n + 1)$

$=$ smallest $x \in X$ s.t. $x \geq x_n + 1$
$\Updownarrow$
$x > x_n$

$= x_{n+1}$ ∎

(d) $X \subseteq \mathbb{N}$ infinite, rec. $\iff X = Rng(f)$ for some $f \in Rec$ incr.

<u>Proof</u> "$\Longrightarrow$" Suppose $X \subseteq \mathbb{N}$ infinite s.t. $\chi_X \in Rec$.

Then construct $\overline{f}$ as in 2(c).

"$\Longleftarrow$" Let $j(n) = \min\{y : f(y) \geq n\}$.

Then $\chi_X(x) = \begin{cases} 1 & \text{if } j(x+1) \neq j(x) \\ 0 & \text{otherwise} \end{cases}$

Why? $j(n) \neq j(n+1)$

$\Updownarrow$

$f(j(n)) < n+1 \land f(j(n+1)) \geq n+1$

Suppose $n \notin X$. Then $f(j(n)) \geq n \iff$

$f(j(n)) > n \iff f(j(n)) \geq n+1$ $\lightning$

because $f(j(n)) \neq n$

On the other hand suppose $j(n) = j(n+1)$.

Then $f(j(n+1)) = \underbrace{f(j(n))}_{\text{smallest element} \geq n \text{ is not } n} \geq n+1$, thus $n \notin X$. ∎

ex. 3

(a) $f \in Rec$. Then

- $Rng(f)$ is recursively enumerable, simply by the definition of recursive enumerability.
- $Dom(f) =: A$. Then

$$g(x,y) = \begin{cases} x & \text{if } f(x) = y \\ 0 & \neg \end{cases}$$

Then $h(x) = g(x, f(x))$ is such that $Rng(x) = A$ (and $Dom(h) = A$).