Zad. 5

c) $y'' + p(t)y' + q(t)y = 0 \qquad (*)$

Wprowadźmy nową zmienną zależną $v(t)$ daną $y(t) = y_1(t)v(t)$

Wtedy $\frac{dy}{dt} = v\frac{dy_1}{dt} + y_1\frac{dv}{dt}$

oraz $\frac{d^2y}{dt^2} = v\frac{d^2y_1}{dt^2} + 2\frac{dv}{dt}\frac{dy_1}{dt} + y_1\frac{d^2}{dt}$

Podstawiając do $(*)$:

$$v\frac{d^2y_1}{dt} + 2\frac{dv}{dt}\frac{dy_1}{dv} + y_1\frac{d^2v}{dt^2}$$

$$+ p(t)\left[v\frac{dy_1}{dt} + y_1\frac{dv}{dt}\right] + q(t)vy_1$$

$$= y_1\frac{d^2v}{dt^2} + \left[2\frac{dy_1}{dt} + p(t)y_1\right]\frac{dv}{dt} +$$

$$+ \underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=0} v =$$

$$= y_1 \frac{d^2 v}{dt^2} + \left[2\frac{dy_1}{dt} + p(t) y_1 \right] \frac{dv}{dt}$$

Zatem $y_1 v$ jest rozw. $(*)$ wtedy, gdy v spełnia

$$y_1 \frac{d^2 v}{dt^2} + \left[2\frac{dy_1}{dt} + p(t) y_1 \right] \frac{dv}{dt} = 0$$

To jest tak naprawdę równanie liniowe pierwszego rzędu zmiennej $\frac{dv}{dt}$. To już umiemy rozwiązywać

$$\frac{dv}{dt} = c \cdot \exp\left(- \int \left[2\frac{y_1'(t)}{y_1(t)} + p(t) \right] dt \right)$$

$$= c \cdot \exp\left(- \int p(t)\, dt \right) \exp\left(- \int 2\frac{y_1'}{y_1} dt \right)$$

$$= \frac{c \exp\left(- \int p(t)\, dt \right)}{y_1^2}$$

Potrzebujemy dowolnego v, więc ustalamy $c = 1$.

Całkując obustronnie po t dostajemy

$$v(t) = \int u(t)\,dt, \quad u(t) = \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}$$

A stąd $y_2(t) = y_1(t)\,v(t) = y_1 \cdot \int u(t)\,dt$

y_2 jest niezależne od y_1.

Gdyby $y_2 = C \cdot y_1$, to $v \equiv C$

a zatem mielibyśmy $\frac{dv}{dt} = 0$,

ale $\frac{dv}{dt} = \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)} \neq 0$.

a) $y = z \exp\left(-\frac{1}{2}\int p(s)\,ds\right)$

$$y' = z'\exp\left(-\frac{1}{2}\int p(s)\,ds\right) +$$

$$+ z\exp\left(-\frac{1}{2}\int p(s)\,ds\right)\cdot\left(-\frac{1}{2}p(s)\right)$$

$$y'' = z''\exp\left(-\frac{1}{2}\int p(s)\,ds\right) + 2z'\exp\left(-\frac{1}{2}\int p(s)\,ds\right)\cdot$$
$$\cdot\left(-\frac{1}{2}p(s)\right)$$

$$+ z\left[\exp\left(-\frac{1}{2}\int p(s)\,ds\right)\left(-\frac{1}{2}p(s)\right)\right]' = (*)$$

$$\parallel$$

$$\exp\left(-\frac{1}{2}\int p(s)\,ds\right)\left(-\frac{1}{2}p(s)\right)^2 +$$
$$+ \exp\left(-\frac{1}{2}\int p(s)\,ds\right)\cdot\left(-\frac{1}{2}p'(s)\right)$$

$$(*) = \exp\left(-\frac{1}{2}\int p(s)\,ds\right)\cdot\left[z'' + 2z'\left(-\frac{1}{2}p(s)\right)\right.$$
$$\left. + z\left(-\frac{1}{2}p(s)\right)^2 + z\left(-\frac{1}{2}p'(s)\right)\right] =$$

$$= \exp\left(-\frac{1}{2}\int p(s)\,ds\right)\left[z'' - z'p(s) + z\frac{1}{4}p^2(s) - \frac{1}{2}zp'(s)\right]$$

$$y = z \cdot E$$

$$y' = z' \cdot E + zE \cdot \left(-\tfrac{1}{2}p\right)$$

$$y'' = z'' \cdot E + z'E \cdot \left(-\tfrac{1}{2}p\right) +$$

$$+ z'\left[E \cdot \left(-\tfrac{1}{2}p\right)\right] + z\left[E \cdot \left(-\tfrac{1}{2}p\right)^2 + E \cdot \left(-\tfrac{1}{2}p'\right)\right] =$$

$$= z''E - z'Ep + \tfrac{1}{4}zEp^2 - \tfrac{1}{2}zEp'$$

$$= E\left(z'' - z'p + \tfrac{1}{4}z\left[p^2 - 2p'\right]\right)$$

$$y'' + py' + qy = E\left(z'' - z'p + \tfrac{1}{4}z\left[p^2 - 2p'\right]\right)$$

$$+ z'pE - \tfrac{1}{2}zEp^2 + zEq =$$

$$= E\Big[z'' - z'p + \tfrac{1}{4}z\left[p^2 - 2p'\right] + z'p$$

$$- \tfrac{1}{2}zp^2 + zq\Big]$$

$$= E\left[z'' + z\left[\tfrac{1}{4}p^2 - \tfrac{1}{2}p^2 - \tfrac{1}{2}p' + q\right]\right] =$$

$$= E\left[z'' + z\left[q - \tfrac{1}{4}p^2 - \tfrac{1}{2}p'\right]\right] = 0$$

Czyli $z'' + z\left(q - \frac{1}{4}p^2 - \frac{1}{2}p'\right) = 0$.

b)