Zad. 3 Niech $Y_i^{(n)} = \begin{cases} 1 & \text{gdy } i\text{-ta urna pusta} \\ 0 & \neg \end{cases}$

Wtedy $X_n = \sum_{k=1}^{n} Y_k^{(n)}$. Ponadto $\mathbb{E}X_n = \sum_{k=1}^{n} \mathbb{E}Y_k^{(n)} =$

$$= n \cdot \underbrace{\left(\frac{n-1}{n}\right)^{k_n}}_{\mathbb{E}Y_k^{(n)} = 1\cdot\mathbb{P}[Y_k^{(n)}=1]} = n\left(1-\frac{1}{n}\right)^{k_n}$$

Zatem $\frac{\mathbb{E}X_n}{n} = \left(1-\frac{1}{n}\right)^{k_n} = \left(1-\frac{1}{n}\right)^{n\cdot\frac{k_n}{n}} \xrightarrow{n\to\infty} e^{-c}$

Chcemy pokazać, że $\frac{X_n - \mathbb{E}X_n}{n} \xrightarrow{\mathbb{P}} 0$.

Wybierzmy $\varepsilon > 0$. Wtedy, z nierówności Czebyszewa

$$\mathbb{P}\left[\frac{X_n-\mathbb{E}X_n}{n} \geq \varepsilon\right] \leq \frac{\operatorname{Var}X_n}{\varepsilon^2 n^2} = \frac{\operatorname{Var}(Y_1^{(n)} + \dots + Y_n^{(n)})}{\varepsilon^2 n^2} = (*)$$

$$\operatorname{Var}(Y_k^{(n)}) = \underbrace{\mathbb{E}(Y_k^{(n)})^2}_{\mathbb{E}Y_k^{(n)}} - (\mathbb{E}Y_k^{(n)})^2 = \mathbb{E}Y_k^{(n)}(1-\mathbb{E}Y_k^{(n)})$$

Dla $i \neq j$ $\operatorname{Cov}(Y_i^{(n)}, Y_j^{(n)}) = \mathbb{E}Y_i^{(n)}Y_j^{(n)} - \mathbb{E}Y_i^{(n)}\mathbb{E}Y_j^{(n)}$

$$= \left(\frac{n-2}{n}\right)^{k_n} - \left(\frac{n-1}{n}\right)^{2k_n} = \left(1-\frac{2}{n}\right)^{k_n} - \left(1-\frac{1}{n}\right)^{2k_n}$$

$$(*) = \frac{1}{\varepsilon^2 n^2}\cdot\left[\sum_{k=1}^{n}\operatorname{Var}Y_k^{(n)} + 2\sum_{i<j}\operatorname{Cov}(Y_i^{(n)}, Y_j^{(n)})\right]$$

$$= \frac{1}{\varepsilon^2}\cdot\left[\underbrace{\frac{1}{n}}_{\to 0}\underbrace{\left(1-\frac{1}{n}\right)^{k_n}}_{\to e^{-c}}\underbrace{\left(1-\left(1-\frac{1}{n}\right)^{k_n}\right)}_{\to 1-e^{-c}} + \underbrace{\left(\frac{n-1}{n}\right)}_{\to 1}\cdot\left(\underbrace{\left(1-\frac{2}{n}\right)^{k_n}}_{\to e^{-2c}} - \underbrace{\left(1-\frac{1}{n}\right)^{2k_n}}_{\to e^{-2c}}\right)\right]$$

$$\xrightarrow{n\to\infty} 0$$

Zatem z definicji $\frac{X_n}{n} \xrightarrow{\mathbb{P}} \frac{\mathbb{E}X_n}{n} = e^{-c}$ ∎