Zad. 10

a) Dla $n = 2k$ mamy

$$\frac{1}{2} \cdot \frac{X_1X_2 + X_3X_4 + \ldots + X_{2k-1}X_{2k}}{k} \xrightarrow[k\to\infty]{} \frac{1}{2}\mathbb{E}X_1X_2 = \frac{1}{2}\lambda^2$$

Podobnie

$$\frac{1}{2}\,\frac{X_2X_3 + \ldots + X_{2k}X_{2k+1}}{k} \xrightarrow[k\to\infty]{} \frac{1}{2}\mathbb{E}X_2X_3 = \frac{1}{2}\lambda^2$$

Zatem $$\frac{X_1X_2 + X_2X_3 + \ldots + X_{2k}X_{2k+1}}{2k} \to \lambda^2$$

Dla $n = 2k+1$

$$\frac{X_1X_2 + \ldots + X_{2k+1}X_{2k+2}}{2k+1} = \frac{1}{2}\,\frac{2k+2}{2k+1} \cdot \frac{X_1X_2 + \ldots + X_{2k+1}X_{2k+2}}{k+1}$$

$$\xrightarrow{k\to\infty} \frac{1}{2}\lambda^2$$

Analogicznie $$\frac{X_2X_3 + \ldots + X_{2k}X_{2k+1}}{2k+1} =$$

$$= \frac{1}{2} \cdot \frac{2k}{2k+1}\,\frac{X_2X_3 + \ldots + X_{2k}X_{2k+1}}{k} \longrightarrow \frac{1}{2}\lambda^2$$

Zatem $\frac{\sum_{i=1}^{n} X_i X_{i+1}}{n} \xrightarrow{n \to \infty} \lambda^2$

b) Całkowicie analogiczne, tylko że

$$\mathbb{E}X_1^2 X_2 = \mathbb{E}X_1^2 \cdot \mathbb{E}X_2$$

$$\mathbb{P}[X^2 = k^2] = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathbb{E}X^2 = \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot k^2 = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k k^2}{k!} =$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!} =$$

$$= \lambda e^{-\lambda} \left[\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \right]$$

$$= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) =$$

$$= \lambda (1 + \lambda) = \lambda + \lambda^2$$

Zatem $\mathbb{E}X_1^2 X_2 = \lambda^3 + \lambda^2$

c) $$\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i X_{i+1}} = \frac{\sum_{i=1}^{n} X_i}{n} \cdot \frac{n}{\sum_{i=1}^{n} X_i X_{i+1}} \longrightarrow \frac{1}{\lambda}$$