Zad. 1

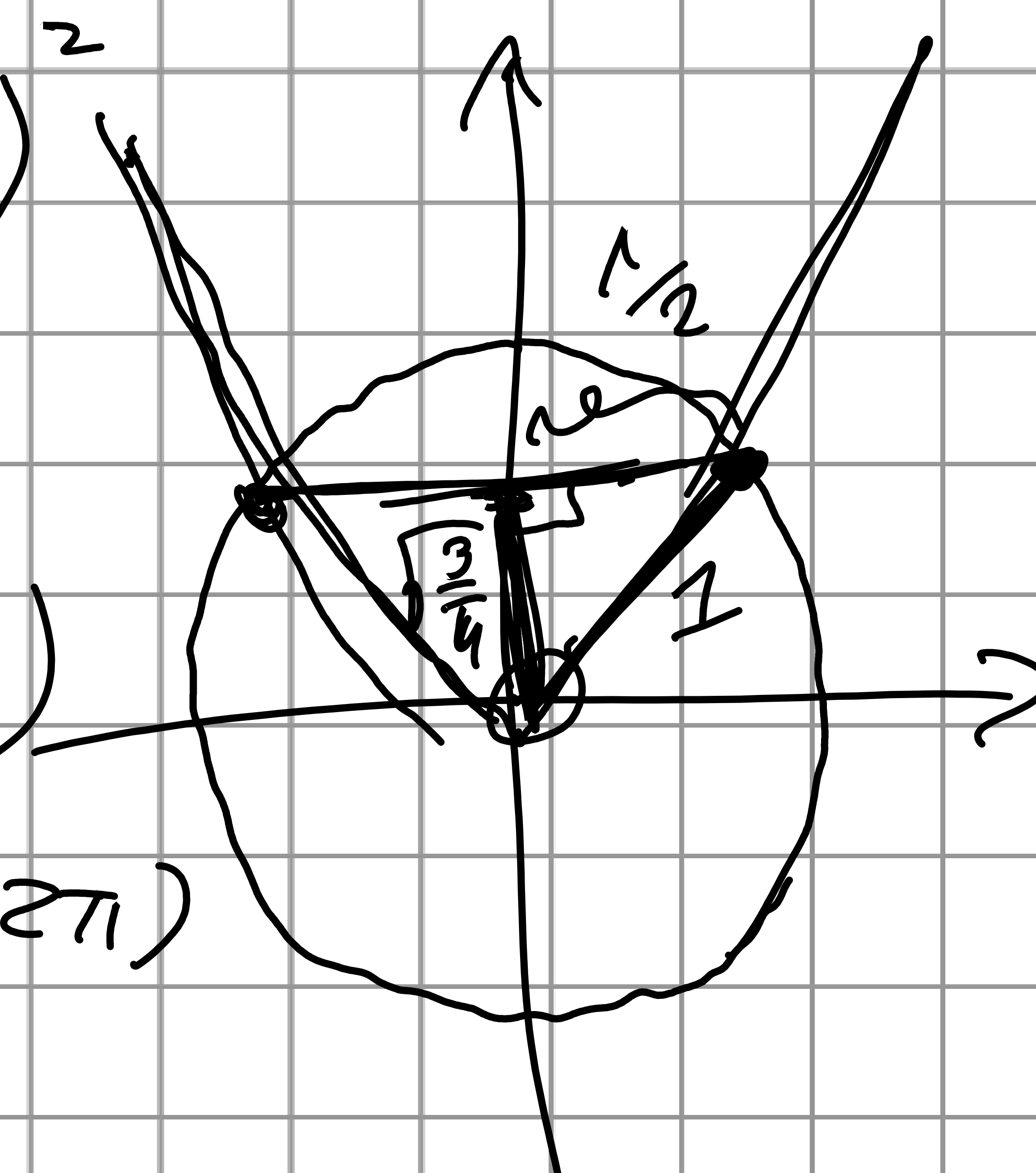
$x^2+y^2+z^2=1$ oraz $z=\sqrt{3}\,(x^2+y^2)^{\frac{1}{2}}$

$$\Downarrow$$

$$x^2+y^2+3(x^2+y^2)=1$$

$$(2x)^2+(2y)^2=1$$

$$x^2+y^2=\left(\frac{1}{2}\right)^2$$




$$\sigma(t)=\left(\frac{1}{2}\cos t,\ \frac{1}{2}\sin t,\ \sqrt{\frac{3}{4}}\right)\qquad t\in[0,2\pi)$$

$$\int_\sigma \rho(x,y,z)\,ds=$$

$$\int_0^{2\pi}\left(\left|\frac{1}{2}\cos t\right|+\left(\frac{1}{2}\sin t\right)^2+\frac{9}{16}\right)\cdot\underbrace{\sqrt{\left(\frac{1}{2}\sin t\right)^2+\left(\frac{1}{2}\cos t\right)^2}}_{=\frac{1}{2}}\,dt=$$

$$=\frac{1}{2}\int_0^{2\pi}\left|\frac{1}{2}\cos t\right|+\left(\frac{1}{2}\sin t\right)^2+\frac{9}{16}\,dt=$$

$$=\frac{1}{2}\cdot\left(4\cdot\int_0^{\frac{\pi}{2}}\frac{1}{2}\cos t\,dt+\frac{1}{4}\int_0^{2\pi}\sin^2 t\,dt+\int_0^{2\pi}\frac{9}{16}\,dt\right)=$$

$$= \frac{1}{2}\left(2 \cdot \left(\sin t\right)\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}\right.$$

$$\left. + \frac{9}{16} t \Big|_0^{2\pi} \right) =$$

$$= \frac{1}{2}\left(2 + \frac{\pi}{2} + \frac{9\pi}{8}\right) =$$

$$= 1 + \frac{13\pi}{16}$$

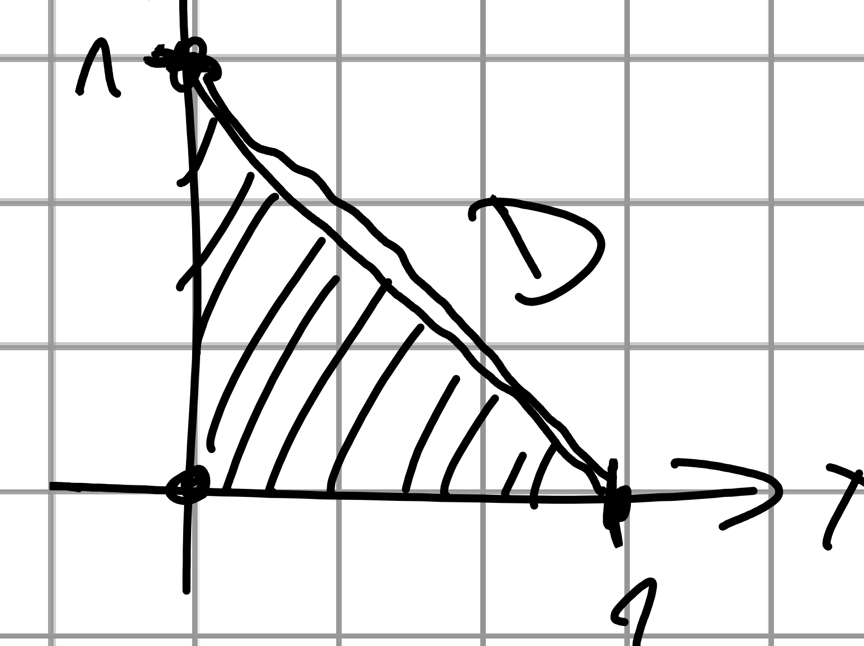
Zad. 2

$$x+y+z=1 \quad , \quad x,y,z \geq 0$$

$$z = 1-x-y$$

$$\Phi(x,y) = (x,\ y,\ 1-x-y)$$

$$x,y \geq 0 \ , \quad x+y \leq 1$$




$$T_x = (1,\ 0,\ -1) \qquad T_x \times T_y = (1,\ 1,\ 1)$$

$$T_y = (0,\ 1,\ -1) \qquad \|T_x \times T_y\| = \sqrt{3}$$

$$\iint_S x^2+2xy \, dS = \int_0^1\int_0^{1-y} (x^2+2xy \cdot)\sqrt{3} \, dx \, dy =$$

$$= \sqrt{3}\int_0^1\int_0^{1-y} x^2+2xy \, dx \, dy =$$

$$= \sqrt{3}\int_0^1 \frac{x^3}{3}+x^2y \Big|_0^{1-y} dy =$$

$$\sqrt{3}\int_0^1 \frac{(1-y)^3}{3} + (1-y)^2 y \, dy =$$

$$= \sqrt{3}\left(-\frac{(1-y)^4}{12}\Big|_0^1 + \int_0^1 y - 2y^2 + y^3 \, dy\right) =$$

$$= \sqrt{3}\left(\frac{1}{12} + \left(\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4}\right)\Big|_0^1\right) =$$

$$= \sqrt{3}\left(\frac{1}{12} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) =$$

$$= \sqrt{3}\left(\frac{1}{12} + \frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right) =$$

$$= \sqrt{3} \cdot \frac{1}{6} = \frac{\sqrt{3}}{6}$$

Zad. 4

$T^2(F(S))$ – przestrzeń liniowa 2-tensorów nad $F(S)$

$F(S\times S)$ – funkcje rzeczywiste nad $S^2$.

Niech $\overset{|S| \, ||}{n} = \dim F(S)$ (S skończone więc można tak napisać)

Wtedy $\dim F(S\times S) = n^2$, podobnie $\dim T^2(F(S)) = n^2$, zatem

$$T^2(F(S)) \cong F(S\times S).$$

Bazy tych przestrzeni to

- $F(S\times S)$: Niech $E_1, \dots, E_n$ – baza $F(S)$. Wtedy

  $E_{ij} \in F(S\times S)$ – baza $F(S^2)$ t. że

  $E_{ij}(e_a \times e_b) = 1 \iff i=a \wedge j=b$,

  gdzie $\{e_1, \dots, e_n\} = S$, $E_i(e_j) = \delta_{ij}$

- $T^2(F(S))$: $\varphi_{ij}$ – baza $T(F(S))$ t. że

  $\varphi_{ij}(E_a, E_b) = 1 \iff i=a \wedge j=b$

Wtedy mamy izomorfizm

$$\Phi : \mathcal{J}^2(F(S)) \longrightarrow F(S^2) \quad \text{dany}$$

na bazie

$$\Phi(\varphi_{ij}) = E_{ij}.$$

Zadanie 5

$f: \mathbb{R}^d \to \mathbb{R}^m$, $f \in C^\infty$.

Najpierw definiuje się

$$f_*: T_p\mathbb{R}^d \longrightarrow T_{f(p)}\mathbb{R}^m$$

która dana jest wzorem

$$f_*((v)_p) = [Df(p)(v)]_{(f(p))}$$

Jest to pochodna $f$, ale w odpowiednich przestrzeniach stycznych.

Wtedy $f^*: T^k(\mathbb{R}^m) \to T^k(\mathbb{R}^d)$

zapisujemy wzorem:

$$\underbrace{f^*(\underset{\in\, T^k(\mathbb{R}^m)}{\omega})(\underset{\in\, \mathbb{R}^d}{p})}_{\in\, \Omega^k(T_p\mathbb{R}^d)}(v_1, \dots, \underset{\in\, T_p\mathbb{R}^d}{v_k}) =$$

$$= \underbrace{\omega(f(p))}_{\in\, \Omega^k(T_{f(p)}\mathbb{R}^m)}(f_*(v_1), \dots, \underset{\in\, T_{f(p)}\mathbb{R}^m}{f_*(v_k)})$$

Teraz $f_1 : \mathbb{R}^n \to \mathbb{R}^m$, $f_2 : \mathbb{R}^d \to \mathbb{R}^n$.

Wtedy

$$\underbrace{(f_1 \circ f_2)^*(\omega)}_{\in T^k(\mathbb{R}^m)}\underbrace{(p)}_{\in \mathbb{R}^d}(v_1, \ldots, \underbrace{v_k}_{\in J_p \mathbb{R}^d}) =$$

$$= \omega((f_1 \circ f_2)(p))((f_1 \circ f_2)_*(v_1), \ldots, (f_1 \circ f_2)_*(v_k)) =$$

$$= \omega((f_1 \circ f_2)(p))((f_{1*} \circ f_{2*})(v_1), \ldots, (f_{1*} \circ f_{2*})(v_k))$$

Ponieważ $(f_1 \circ f_2)_*(x_p) = [D(f_1 \circ f_2)(p)(x_p)]_{(f_1 \circ f_2)(p)} =$

$= Df_1(f_2(p)) \cdot Df_2(p)(x_p) = (f_{1*} \circ f_{2*})(x_p)$

Z drugiej strony

$$(\underbrace{f_2^*}_{T^k(\mathbb{R}^n) \to T^k(\mathbb{R}^d)} \circ \underbrace{f_1^*}_{T^k(\mathbb{R}^m) \to T^k(\mathbb{R}^n)})(\omega)(p)(v_1, \ldots, v_k) =$$

$$= f_2^*(f_1^*(\omega))(p)(v_1, \ldots, v_k) =$$

$$= f_1^*(\omega)(f_2(p))(f_{2*}(v_1), \ldots, f_{2*}(v_k)) =$$

$$= \omega((f_1 \circ f_2)(p))((f_{1*} \circ f_{2*})(v_1), \ldots, (f_{1*} \circ f_{2*})(v_k))$$