Zad. 12.3

$$\langle f, g \rangle = \int_0^1 (1+x^2) f(x) g(x)\, dx$$

$$A_0 = \int_0^1 (1+x^2) \frac{x - x_1}{x_0 - x_1}\, dx,$$

$$A_1 = \int_0^1 (1+x^2) \frac{x - x_0}{x_1 - x_0}\, dx$$

$$P_0(x) = 1, \quad P_1(x) = \left(x - \frac{\langle xP_0, P_0\rangle}{\langle P_0, P_0\rangle}\right) P_0(x)$$

$$P_2(x) = \left(x - \frac{\langle xP_1, P_1\rangle}{\langle P_1, P_1\rangle}\right) P_1(x) - \frac{\langle P_1, P_1\rangle}{\langle P_0, P_0\rangle}$$

$$\langle P_0, P_0\rangle = \int_0^1 (1+x^2)\, dx = \frac{4}{3}$$

$$\longrightarrow P_1(x) = x - \frac{9}{16}$$

$$\langle xP_0, P_0\rangle = \int_0^1 (x + x^3)\, dx = \frac{3}{4}$$

$$\langle P_1, P_1\rangle = \int_0^1 (1+x^2)\left(x - \frac{9}{16}\right)^2 dx = \frac{107}{960}$$

$$\langle xP_1, P_1\rangle = \int_0^1 x(1+x^2)\left(x - \frac{9}{16}\right)^2 dx = \frac{829}{15360}$$

$$\frac{795\,840}{1643520} = \frac{829}{1712} = \frac{\langle xP_1, P_1\rangle}{\langle P_1, P_1\rangle}, \quad \frac{\langle P_1, P_1\rangle}{\langle P_0, P_0\rangle} = \frac{107}{1280}$$

$$P_2(x) = \left(x - \frac{829}{1712}\right)\left(x - \frac{9}{16}\right) - \frac{107}{1280}$$

$$X_0 =$$

zad. 12.4

$W_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, dowolny

Q_i – wiel. ortogonalny

$Q_n(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_0$

$$W_n(x) = Q_n(x) + \underbrace{R(x)}_{\in \Pi_{n-1}}$$

$$\int_a^b p(x) W_n^2(x)\,dx = \int_a^b p(x)\left[Q_n(x) + R(x)\right]^2 dx =$$

$$= \int_a^b p(x) Q_n^2(x)\,dx + 2\underbrace{\int_a^b p(x) Q_n(x) R(x)\,dx}_{= 0} +$$

$$+ \int_a^b p(x) R^2(x)\,dx =$$

$$= \underbrace{\langle Q_n, Q_n \rangle}_{> 0} + \underbrace{\langle R, R \rangle}_{> 0} \geq \langle Q_n, Q_n \rangle$$

zad. 12.7

$$y'(t) = \lambda y(t) \quad (t > 0), \quad y(0) = 1$$
$$\lambda < 0$$

$$y'(t) = f(t, y(t))$$
$$y_{n+1} = y_n + h f_n \quad \text{(jawne)}$$
$$y_{n+1} = y_n + h f_{n+1} \quad \text{(niejawne)}$$

$$f(t, y(t)) = \lambda y(t) \qquad f_k = f(t_k, y_k)$$

jawna: $$y_{n+1} = y_n + h \cdot f(t_n, \overbrace{y(t_n)}^{y_n})$$
$$y_{n+1} = y_n + h\lambda y_n$$
$$y_{n+1} = (1 + h\lambda) y_n$$

$$y_n \to 0 \iff |1 + h\lambda| < 1$$
$$\Downarrow$$
$$\frac{-2}{\lambda} > h > 0$$

niejawna: $$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$$
$$= y_n + h\lambda y_{n+1}$$
$$\Downarrow$$
$$y_{n+1} = \frac{y_n}{1 - h\lambda}$$

$$-1 < \frac{1}{1 - h\lambda} < 1$$

Gdy h: $\quad h \neq \frac{1}{\lambda}$

Ponadto

1° $\quad 1 - h\lambda > 0 \Rightarrow 1 - h\lambda > 1$

$$\Updownarrow \qquad\qquad \Downarrow$$

$$1 > h\lambda \qquad\qquad 0 > h\lambda^{<0}$$

$$\frac{1}{\lambda} < h \qquad\qquad \Downarrow$$

$$h > 0$$

2° $\quad 1 - h\lambda < 0 \Rightarrow h\lambda - 1 > 1$

$$\Updownarrow \qquad\qquad h\lambda > 2$$

$$\frac{1}{\lambda} > h \qquad\qquad h < \frac{2}{\lambda}$$

$$\Downarrow$$

$$h \in (0, \infty) \cup \left(-\infty, \frac{2}{\lambda}\right)$$

zad. 12.8

$$y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1}) =$$

$$= y_n + \frac{h}{2}(\lambda y_n + \lambda y_{n+1})$$

$$y_{n+1} = \frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda} y_n$$

zad. 12.5

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}$$

$$(1)\quad \begin{cases} x'(t) = u(t) \\ u'(t) = -z(t)u(t) \\ y'(t) = v(t) \\ v'(t) = -g - z(t)v(t) \end{cases}$$

$$(2)\quad \begin{cases} x_{n+1} = x_n + h u_n \\ u_{n+1} = u_n - h z(t_n) u_n \\ \qquad = u_n - hk\sqrt{u_n^2 + v_n^2}\, u_n = u_n\left(1 - hk\sqrt{u_n^2+v_n^2}\right) \\ y_{n+1} = y_n + h v_n \\ v_{n+1} = v_n - h\left(g + z(t_n) v_n\right) = \\ \qquad = v_n - h\left(g + k\sqrt{u_n^2+v_n^2}\, v_n\right) \end{cases}$$

RESZTA NA KOMPUTERZE