LISTA 9

Zad. 3 4 PKT

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \lim_{a'\to a^+}\int_{a'}^b f(\varphi(x))\varphi'(x)\,dx = \left|\begin{matrix} u = \varphi(x) \\ du = \varphi'(x)\,dx \end{matrix}\right| =$$

$$= \lim_{a'\to a^+}\int_{\varphi(a')}^{\varphi(b)} f(u)\,du = \lim \int_{\varphi(a)}^{\varphi(b)} f(u)\,du$$

bo φ jest ciągłe na $[a,b]$ więc przyjmuje wszystkie wartości między $\varphi(a)$, $\varphi(b)$ więc f jest ciągłe na przedziale $[\varphi(a), \varphi(b)]$.

Zad. 5 4 PKT

Załóżmy że φ jest ciągła i różniczkowalna na $[a, +\infty)$ oraz f jest ciągła we wszystkich wartościach φ.

$$T: \int_a^{+\infty} f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(a)}^{L} f(u)\,du \quad , \text{gdzie } L = \lim_{x\to\infty}\varphi(x).$$

D-d.

$$\int_a^{+\infty} f(\varphi(x))\varphi'(x)\,dx = \lim_{b\to\infty}\int_a^b f(\varphi(x))\varphi'(x)\,dx = \left|\begin{matrix} u = \varphi(x) \\ du = \varphi'(x)\,dx \end{matrix}\right| =$$

$$= \lim_{b\to+\infty}\int_{\varphi(a)}^{\varphi(b)} f(u)\,du = \int_{\varphi(a)}^{L} f(u)\,du.$$

bo f ciągła na wszystkich wartościach φ, więc ciągła w przedziale $[\varphi(a), L)$.

zad. 6 2 PKT

$$\int_0^\infty \cos x^2\,dx = \lim_{b\to\infty}\int_0^{b} \cos x^2\,dx = \left|\begin{array}{l} \sqrt{u} = x \\ \frac{1}{2\sqrt{u}}\,du = dx \end{array}\right| =$$

$$= \lim_{b\to\infty}\int_0^{\sqrt{b}} \frac{\cos u}{2\sqrt{u}}\,du = \frac{1}{2}\int_0^\infty \frac{\cos u}{\sqrt{u}}\,du .$$

zad. 8 5 PKT

a) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$. Niech $\varepsilon > 0$, $b_0 = \frac{4}{\varepsilon^2}$, wtedy dla $b_1, b_2 > b_0$ mamy

$$\int_{b_1}^{b_2} \frac{\sin x}{\sqrt{x}}\,dx = \frac{1}{\sqrt{b_1}}\int_{b_1}^{\xi} \sin x\,dx = \frac{1}{\sqrt{b_1}}\left(\cos b_1 - \underset{\leq 2}{\cos \xi}\right) \leq \frac{2}{\sqrt{b_1}} < \varepsilon$$

Tw. o wartości średniej.

Z drugiej strony

$$\int_0^\infty \frac{|\sin x|}{\sqrt{x}}\,dx : \int_{k\pi}^{k\pi+\pi} \frac{|\sin x|}{\sqrt{x}}\,dx = \frac{1}{\sqrt{k\pi}}\int_{k\pi}^{\xi} |\sin x|\,dx \geq$$

$$\geq \frac{1}{\sqrt{k\pi+\pi}} \cdot \int_{k\pi}^{k\pi+\pi} |\sin x|\,dx = \frac{2}{\sqrt{\pi}\cdot\sqrt{k+1}}$$

$$\int_0^\infty \frac{|\sin x|}{\sqrt{x}}\,dx = \sum_{k=0}^\infty \int_{k\pi}^{k\pi+\pi} \frac{|\sin x|}{\sqrt{x}}\,dx \geq \sum_{k=0}^\infty \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{k+1}} \longrightarrow +\infty .$$

b) $\int\limits_0^\infty \frac{\cos x}{\sqrt{x}(1+x)} dx$. To jest bezwzględnie zbieżne (co impliku zbieżność). Weźmy $\varepsilon > 0$. Niech $b_0 = \frac{4}{\varepsilon^2}$. Wtedy jeśli $b_1, b_2 > b_0$

$$\int_{b_1}^{b_2} \frac{|\cos x|}{\sqrt{x}(1+x)\cancel{dx}}\,dx \leq \int_{b_1}^{b_2} \frac{1}{\sqrt{x}(1+x)}\,dx \leq \int_{b_1}^{b_2} \frac{1}{x\sqrt{x}}\,dx =$$

$$= -\frac{2}{\cancel{x}\sqrt{b_2}} - \left(-\frac{2}{2\sqrt{b_1}}\right) = 2\left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) \leq$$

$$\leq 2\frac{1}{\sqrt{b_1}} < \frac{2}{\sqrt{b_0}} = \frac{2}{\sqrt{\frac{4}{\varepsilon^2}}} = \varepsilon$$

c) $\displaystyle\int_{\pi}^{\infty} \frac{\cos x \, dx}{x+\sin 2x}$. Weźmy $\varepsilon > 0$. Niech $b_0 = \max\left(\frac{2}{\varepsilon}+1, \pi\right)$

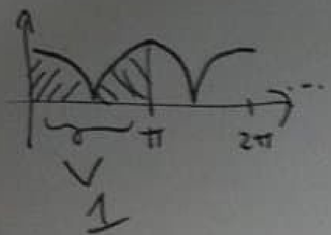
Wtedy dla $b_1, b_2 > b_0$

$$\int_{b_1}^{b_2} \frac{\cos x}{x+\sin 2x}\,dx \leq \int_{b_1}^{b_2} \frac{\cos x}{x-1}\,dx = \frac{1}{b_1-1}\int_{b_1}^{\xi} \cos x \, dx \cancel{\underset{\Delta}{=}} =$$

$$\cancel{\underset{\Delta}{=}} = \frac{\sin \xi - \sin b_1}{b_1 - 1} \leq \frac{2}{b_1-1} < \frac{2}{b_0-1} = \varepsilon$$

Z drugiej strony

$$\int_{k\pi}^{k\pi+\pi} \left|\frac{\cos x}{x+\sin 2x}\right| dx \geq \int_{k\pi}^{k\pi+\pi} \frac{|\cos x|}{x+1}\,dx \geq \frac{1}{k\pi+\pi+1}\int_{k\pi}^{k\pi+\pi} |\cos x|\,dx \geq$$



$$\geq \frac{1}{k\pi+\pi+1}.$$

$$\int_{\pi}^{\infty} \left|\frac{\cos x}{x+\sin 2x}\right| dx \geq \sum_{k=1}^{\infty} \frac{1}{k\pi+\pi+1} \longrightarrow \infty.$$

Zad. 12 10PKT

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^\alpha)} = \int_0^1 \frac{dx}{(1+x^2)(1+x^\alpha)} + \int_1^\infty \frac{dx}{(1+x^2)(1+x^\alpha)} = \left| \begin{array}{l} x = \frac{1}{u} \\ dx = -\frac{1}{u^2} du \end{array} \right| =$$

$$= \int_\infty^1 - \frac{\frac{1}{u^2}\, du}{\left(1+\frac{1}{u^2}\right)\left(1+\frac{1}{u^\alpha}\right)} + \int_1^\infty \frac{dx}{(1+x^2)(1+x^\alpha)} =$$

$$= \int_1^\infty \frac{1}{x^2\left(1+\frac{1}{x^2}\right)\left(1+\frac{1}{x^\alpha}\right)} + \frac{1}{(1+x^2)(1+x^\alpha)}\, dx =$$

$$= \int_1^\infty \frac{1}{(1+x^2)\left(1+\frac{1}{x^\alpha}\right)} + \frac{1}{(1+x^2)(1+x^\alpha)}\, dx = \int_1^\infty \frac{2 + \frac{1}{x^\alpha} + x^\alpha}{(1+x^2)(1+x^\alpha)\left(1+\frac{1}{x^\alpha}\right)}\, dx =$$

$$= \int_1^\infty \frac{dx}{(1+x^2)} = \arctan(x) \Big|_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$