Zad* 26.03

$$\int_0^1 (1-x^4)^n dx = \underbrace{x(1-x^4)^n \Big|_0^1}_{0} - \int_0^1 n(1-x^4)^{n-1} x\cdot(-4x^3)dx =$$

$$= 4n\int_0^1 (1-x^4)^{n-1} x^4 dx =$$

$$= 4n\left[ \underbrace{\frac{x^5}{5}(1-x^4)^{n-1}\Big|_0^1}_{0} - \int_0^1 \frac{x^5}{5}(n-1)(1-x^4)^{n-2}(-4x^3)dx\right] =$$

$$= \frac{4^2 n(n-1)}{5}\cdot\int_0^1 x^8 (1-x^4)^{n-2} dx = \ldots =$$

$$= \frac{4^{n-1}n(n-1)\ldots 2}{5\cdot 9\cdot\ldots\cdot(4n-7)}\cdot\int_0^1 x^{4n-4}(1-x^4)^1 dx =$$

$$= \frac{4^{n-1}n(n-1)\ldots 2}{5\cdot 9\cdot\ldots\cdot(4n-7)}\left(\frac{x^{4n-3}}{4n-3}(1-x^4)\Big|_0^1 - \int_0^1 \frac{x^{4n-3}}{4n-3}\cdot(-4x^3)dx\right) =$$

$$= \frac{4^n n(n-1)\ldots 2}{5\cdot 9\cdot\ldots\cdot(4n-7)(4n-3)}\cdot\int_0^1 x^{4n} dx =$$

$$= \frac{4^n n!}{5\cdot 9\cdot\ldots\cdot(4n-7)(4n-3)(4n+1)}$$

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$$\int_0^1 x^{-x}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x\log x)^n}{n!}dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-x\log x)^n}{n!}dx$$

$$\int_0^1 \frac{(-x\log x)^n}{n!}dx = (-1)^n \frac{1}{n!} \int_0^1 (x\log x)^n dx$$

$$\int_0^1 (x\log x)^n dx = \frac{x^{n+1}}{n+1}\log^n x \Big|_0^1 - \frac{1}{n+1}\int_0^1 x^{n+1}\cdot n\log^{n-1}x\cdot\frac{1}{x}dx =$$

$$\lim_{x\to 0} \log^n x\cdot x^{n+1} = 0$$

$$= -\frac{n}{n+1}\int_0^1 x^n \log^{n-1}x\, dx =$$

$$= -\frac{n}{n+1}\left[\underbrace{\frac{x^{n+1}}{n+1}\log^{n-1}x \Big|_0^1}_{=0} - \int_0^1 \frac{x^{n+1}}{n+1}(n-1)\log^{n-2}x\cdot\frac{1}{x}dx\right] =$$

$$= (-1)^2\frac{n(n-1)}{(n+1)^2}\int_0^1 x^n \log^{n-2}x\, dx = \ldots =$$

$$= (-1)^{n-1}\frac{n(n-1)\ldots 2}{(n+1)^{n-1}}\int_0^1 x^n\log x\, dx =$$

$$= (-1)^{n-1}\frac{n(n-1)\ldots 2}{(n+1)^{n-1}}\left[\frac{x^{n+1}}{n+1}\log x\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx\right] =$$

$$= (-1)^n\frac{n!}{(n+1)^n}\cdot\int_0^1 x^n dx = (-1)^n\frac{n!}{(n+1)^n \cdot (n+1)}x^{n+1}\Big|_0^1 =$$

$$= (-1)^n\frac{n!}{(n+1)^{n+1}}$$

Zatem

$$\int_0^1 \frac{(-x\log x)^n}{n!}\,dx = \frac{(-1)^n}{n!}\cdot\int_0^1 (x\log x)^n\,dx = (n+1)^{-(n+1)}$$

Stąd

$$\sum_{n=0}^{\infty}\int_0^1 \frac{(-x\log x)^n}{n!}\,dx = \sum_{n=1}^{\infty} n^{-n} \quad \blacksquare$$