zad. 1

$$A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}, \quad A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \Longrightarrow a = -\frac{1}{2}, \; c = 0$$

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Zatem $A = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$

zad. 3

a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, A diagonalna więc $\lambda_1 = 2$, $\lambda_2 = 3$

$$\mu_A(t) = (t-2)(t-3)$$

$$V_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad V_2 = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

b) $\begin{vmatrix} t-2 & 1 \\ 1 & t-1 \end{vmatrix} = (t-1)(t-2) - 1 = t^2 - 3t + 1 = \mu_B(t)$

$$t_0 = \frac{3 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{3+\sqrt{5}}{2}, \quad \lambda_2 = \frac{3-\sqrt{5}}{2}$$

Wektory własne:

1° $\begin{cases} (2-\lambda_1)x + y = 0 \\ x + (1-\lambda_1)y = 0 \end{cases}$, np. $x = 1$, $y = \lambda_1 - 2$

2° to samo, tylko z lambdą dwa.

$$V_1 = \left\{ \alpha \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}, \quad V_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -\frac{\sqrt{5}-1}{2} \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

Zad. 6 Zał. że A jest macierzą górnotrójkątną o parami różnymi wartościami na przekątnej.

$$\det(tE-A) = \begin{vmatrix} t-\lambda_1 & & & * \\ & t-\lambda_2 & & \\ & & \ddots & \\ 0 & & & t-\lambda_n \end{vmatrix} = (t-\lambda_1)\ldots(t-\lambda_n).$$

Stąd $\operatorname{Spec} A = \{\lambda_1, \ldots, \lambda_n\}$.

Z def. 6 widmo A jest proste, a z tw. 5 A jest diagonalizowalna.

zad. 7 a) Zał. $P_1, \ldots, P_n$ lnz. $P=[P_1;\ldots;P_n]$, $D=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$

$A \cdot P_i = \lambda_i P_i$, stąd

$$PDP^{-1} = [\lambda_1 P_1; \lambda_2 P_2; \ldots; \lambda_n P_n]\, P^{n-1} =$$

$$= [AP_1; AP_2; \ldots; AP_n] \cdot P^{n-1} = A \cdot P \cdot P^{n-1} = A$$

b) Zał. że P odwracalne, D diagonalne

$A = PDP^{-1} \Rightarrow A \cdot P = P \cdot D$, jeśli przez $P_1, \ldots, P_n$ oznaczymy kolumny P, to wtedy

$$[AP_1; \ldots; AP_n] = A \cdot P = P \cdot D = [\lambda_1 P_1; \ldots; \lambda_n P_n] \Rightarrow \text{teza.}$$

zad. 11

Pokażemy indukcyjnie, że $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$

Podstawa $n=1$ oczywista.

Załóżmy teraz dla n. Wtedy

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^{n+1} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \overset{\text{ZI}}{=}$$

$$= \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \overset{\binom{n}{1}+\binom{n}{2}=\binom{n+1}{2}}{=}$$

$$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n & \binom{n+1}{2}\lambda^{n-1} \\ 0 & \lambda^{n+1} & (n+1)\lambda^n \\ 0 & 0 & \lambda^{n+1} \end{pmatrix}$$

Zad. 21

$$\begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} X_{k+n-1} \\ X_{k+n-2} \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} X_{k+n} \\ X_{k+n-1} \\ \vdots \\ X_{k+1} \end{bmatrix}$$

$$\underset{A}{\overset{\|}{}} \qquad \underset{X_k}{\overset{\|}{}} \qquad \underset{X_{k+1}}{\overset{\|}{}}$$

Stąd $A^m X_k = A^{m-1}(AX_k) = \ldots =$

$= A(A(\ldots(AX_k))\ldots)) = X_{k+m}$

$\chi_A(t) = t^n - a_{n-1}t^{n-1} - \ldots - a_1 t - a_0$

$\Downarrow$

$\lambda_1, \ldots, \lambda_n$ to wartości własne.

$$\begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_i^{n-1} \\ \lambda_i^{n-2} \\ \vdots \\ \lambda_i^{0} \end{bmatrix} = \begin{bmatrix} \lambda_i^{n} \\ \lambda_i^{n-1} \\ \vdots \\ \lambda_i^{1} \end{bmatrix},$$ zatem

$\begin{bmatrix} \lambda_i^{n-1} \\ \vdots \\ \lambda_i^{0} \end{bmatrix}$ to wektory własne A. zad. 7

Niech
$$P = \begin{bmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

Wtedy z zad. 7

$$A = P D \cdot P^{-1}, \text{ gdzie } D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Niech $P^{-1} = (p_{ij})$.

Niech $P^{-1} = (p_{ij})$.

Wtedy $A^k \cdot X_0 = X_k$, ale z drugiej strony

$$A^k \cdot X_0 = P D^k P^{-1} X_0 =$$

$$= \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \\ \vdots & & \vdots \\ \lambda_1^0 & \cdots & \lambda_n^0 \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k P^{-1} X_0 =$$

$$= \begin{bmatrix} \lambda_1^{n-1+k} & \cdots & \lambda_n^{n-1+k} \\ \vdots & & \vdots \\ \lambda_1^k & \cdots & \lambda_n^k \end{bmatrix} P^{-1} \begin{bmatrix} X_{n-1} \\ \vdots \\ X_0 \end{bmatrix} = \begin{bmatrix} X_{n-1+k} \\ \vdots \\ X_k \end{bmatrix}$$

Stąd

$$X_k = [\lambda_1^k \dots \lambda_n^k] \cdot P^{-1} \cdot \begin{bmatrix} X_{n-1} \\ \vdots \\ X_0 \end{bmatrix} =$$

Stałe i liniowe

$$= c_1 \lambda_1^k + c_2 \lambda_2^k + \dots + c_n \lambda_n^k$$

Bo P^{-1} ustalone, tak samo $X_{n-1}, \dots, X_0$ ∎