## 1. Introduction

## 2. Preliminaries

### 2.1. Descriptive set theory.

**Definition 2.1.** Suppose $X$ is a topological space and $A \subseteq X$. We say that $A$ is *meagre* in $X$ if $A = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n$ are nowhere dense subsets of $X$ (i.e. $\mathrm{Int}(\bar{A_n}) = \emptyset$).

**Definition 2.2.** We say that $A$ is *comeagre* in $X$ if it is a complement of a meager set. Equivalently, a set is comeagre iff it contains a countable intersection of open dense sets.

Every countable set is nowhere dense in any $T_1$ space, so, for example, $\mathbb{Q}$ is meager in $\mathbb{R}$ (though being dense), which means that the set of irrationals is comeagre. Another example is...

**Definition 2.3.** We say that a topological space $X$ is a *Baire space* if every comeagre subset of $X$ is dense in $X$ (equivalently, every meagre set has empty interior).

**Definition 2.4.** Suppose $X$ is a Baire space. We say that a property $P$ *holds generically* for a point in $x \in X$ if $\{x \in X \mid P \text{ holds for } x\}$ is comeagre in $X$.

**Definition 2.5.** Let $X$ be a nonempty topological space and let $A \subseteq X$. The *Banach-Mazur game of* $A$, denoted as $G^{\star\star}(A)$ is defined as follows: Players $I$ and $II$ take turns in playing nonempty open sets $U_0, V_0, U_1, V_1, \ldots$ such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. We say that player $II$ wins the game if $\bigcap_n V_n \subseteq A$.

There is an important theorem on the Banach-Mazur game: $A$ is comeagre iff $II$ can always choose sets $V_0, V_1, \ldots$ such that it wins. Before we prove it we need to define notions necessary to formalize this theorem.

**Definition 2.6.** $T$ is *the tree of all legal positions* in the Banach-Mazur game $G^{\star\star}(A)$ when $T$ consists of all finite sequences $(W_0, W_1, \ldots, W_n)$, where $W_i$ are nonempty open sets such that $W_0 \supseteq W_1 \supseteq \ldots \supseteq W_n$. In another words, $T$ is a pruned tree on $\{W \subseteq X \mid W \text{is open nonempty}\}$.

By $[T]$ we denote the set of all "infinite branches" of $T$, i.e. infinite sequences $(U_0, V_0, \ldots)$ such that $(U_0, V_0, \ldots U_n, V_n) \in T$ for any $n \in \mathbb{N}$.

**Definition 2.7.** A *strategy* for $II$ in $G^{\star\star}(A)$ is a subtree $\sigma \subseteq T$ such that

(i) $\sigma$ is nonempty,
(ii) if $(U_0, V_0, \ldots, V_n) \in \sigma$, then for all open nonempty $U_{n+1} \subseteq V_n$, $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma$,
(iii) if $(U_0, V_0, \ldots, U_n) \in \sigma$, then for unique $V_n$, $(U_0, V_0, \ldots, U_n, V_n) \in \sigma$.

Intuitively, the strategy $\sigma$ works as follows: $I$ starts playing $U_0$ as any open subset of $X$, then $II$ plays unique (by (iii)) $V_0$ such that $(U_0, V_0) \in \sigma$. Then $I$ responds by playing any $U_1 \subseteq V_0$ and $II$ plays uniqe $V_1$ such that $(U_0, V_0, U_1, V_1) \in \sigma$, etc.

### 2.2. Fraïssé classes.

**Fact 2.8** (Fraïssé theorem)**.** *Then there exists a unique up to isomorphism counable $L$-structure $M$ such that...*

**Definition 2.9.** For $\mathscr{C}$, $M$ as in Fact 2.8, we write $\mathrm{FLim}(\mathscr{C}) := M$.

**Fact 2.10.** *If $\mathscr{C}$ is a uniformly locally finite Fraïssé class, then* $\mathrm{FLim}(\mathscr{C})$ *is $\aleph_0$-categorical and has quantifier elimination.*

## 3. Conjugacy classes in automorphism groups

3.1. **Prototype: pure set.** In this section, $M = (M, =)$ is an infinite countable set (with no structure beyond equality).

**Proposition 3.1.** *If $f_1, f_2 \in \mathrm{Aut}(M)$, then $f_1$ and $f_2$ are conjugate if and only if for each $n \in \mathbb{N} \cup \{\aleph_0\}$, $f_1$ and $f_2$ have the same number of orbits of size $n$.*

**Proposition 3.2.** *The conjugacy class of $f \in \mathrm{Aut}(M)$ is dense if and only if...*

**Proposition 3.3.** *If $f \in \mathrm{Aut}(M)$ has an infinite orbit, then the conjugacy class of $f$ is meagre.*

**Proposition 3.4.** *An automorphism $f$ of $M$ is generic if and only if...*

*Proof.* □

3.2. **More general structures.**

**Proposition 3.5.** *Suppose $M$ is an arbitrary structure and $f_1, f_2 \in \mathrm{Aut}(M)$. Then $f_1$ and $f_2$ are conjugate if and only if $(M, f_1) \cong (M, f_2)$.*

**Definition 3.6.** We say that a Fraïssé class $\mathscr{C}$ has *weak Hrushovski property* (*WHP*) if for every $A \in \mathscr{C}$ and partial automorphism $p : A \to A$, there is some $B \in \mathscr{C}$ such that $p$ can be extended to an automorphism of $B$, i.e. there is an embedding $i : A \to B$ and a $\bar{p} \in \mathrm{Aut}(B)$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\bar{p}} & B \\ {\scriptstyle i}\uparrow & & {\scriptstyle i}\uparrow \\ A & \xrightarrow{p} & A \end{array}$$

**Proposition 3.7.** *Suppose $\mathscr{C}$ is a Fraïssé class in a relational language with WHP. Then generically, for an $f \in \mathrm{Aut}(\mathrm{FLim}(\mathscr{C}))$, all orbits of $f$ are finite.*

**Proposition 3.8.** *Suppose $\mathscr{C}$ is a Fraïssé class in an arbitrary countable language with WHP. Then generically, for an $f \in \mathrm{Aut}(\mathrm{FLim}(\mathscr{C}))$ ...*

3.3. **Random graph.**

**Definition 3.9.** The *random graph* is...

**Fact 3.10.** *The*

**Proposition 3.11.** *Generically, the set of fixed points of $f \in \mathrm{Aut}(M)$ is isomorphic to $M$ (as a graph).*