1. Introduction

2. Preliminaries

2.1. Descriptive set theory.

Definition 2.1. Suppose X is a topological space and $A \subseteq X$. We say that A is *meagre* in X if $A = \bigcup_{n \in \mathbb{N}} A_n$, where A_n are nowhere dense subsets of X (i.e. $\mathrm{Int}(\bar{A_n}) = \emptyset$).

Definition 2.2. We say that A is *comeagre* in X if it is a complement of a meagre set. Equivalently, a set is comeagre iff it contains a countable intersection of open dense sets.

Every countable set is meagre in any T_1 space, so, for example, $\mathbb{Q}$ is meagre in $\mathbb{R}$ (though being dense), which means that the set of irrationals is comeagre. Another example is...

Definition 2.3. We say that a topological space X is a *Baire space* if every comeagre subset of X is dense in X (equivalently, every meagre set has empty interior).

Definition 2.4. Suppose X is a Baire space. We say that a property P *holds generically* for a point in $x \in X$ if $\{x \in X \mid P \text{ holds for } x\}$ is comeagre in X.

Definition 2.5. Let X be a nonempty topological space and let $A \subseteq X$. The *Banach-Mazur game of* A, denoted as $G^{\star\star}(A)$ is defined as follows: Players I and II take turns in playing nonempty open sets $U_0, V_0, U_1, V_1, \dots$ such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$. We say that player II wins the game if $\bigcap_n V_n \subseteq A$.

There is an important theorem on the Banach-Mazur game: A is comeagre iff II can always choose sets $V_0, V_1, \dots$ such that it wins. Before we prove it we need to define notions necessary to formalise and prove the theorem.

Definition 2.6. T is *the tree of all legal positions* in the Banach-Mazur game $G^{\star\star}(A)$ when T consists of all finite sequences $(W_0, W_1, \dots, W_n)$, where W_i are nonempty open sets such that $W_0 \supseteq W_1 \supseteq \dots \supseteq W_n$. In another words, T is a pruned tree on $\{W \subseteq X \mid W \text{is open nonempty}\}$.

Definition 2.7. We say that σ is *a pruned subtree* of the tree of all legal positions T if $\sigma \subseteq T$ and for any $(W_0, W_1, \dots, W_n) \in \sigma, n \geq 0$ there is a W such that $(W_0, W_1, \dots, W_n, W) \in \sigma$ (it simply means that there's no finite branch in σ).

Definition 2.8. Let σ be a pruned subtree of the tree of all legal positions T. By $[\sigma]$ we denote *the set of all infinite branches of* σ, i.e. infinite sequences $(W_0, W_1, \dots)$ such that $(W_0, W_1, \dots W_n) \in \sigma$ for any $n \in \mathbb{N}$.

Definition 2.9. A *strategy* for II in $G^{\star\star}(A)$ is a pruned subtree $\sigma \subseteq T$ such that

(i) σ is nonempty,
(ii) if $(U_0, V_0, \dots, U_n, V_n) \in \sigma$, then for all open nonempty $U_{n+1} \subseteq V_n$, $(U_0, V_0, \dots, U_n, V_n, U_{n+1}) \in \sigma$,
(iii) if $(U_0, V_0, \dots, U_n) \in \sigma$, then for a unique V_n, $(U_0, V_0, \dots, U_n, V_n) \in \sigma$.

Intuitively, a strategy σ works as follows: I starts playing U_0 as any open subset of X, then II plays unique (by (iii)) V_0 such that $(U_0, V_0) \in \sigma$. Then I responds by playing any $U_1 \subseteq V_0$ and II plays uniqe V_1 such that $(U_0, V_0, U_1, V_1) \in \sigma$, etc.

Definition 2.10. A strategy σ is a *winning strategy for II* if for any game $(U_0, V_0 \ldots) \in [\sigma]$ player II wins, i.e. $\bigcap_n V_n \subseteq A$.

Now we can state the key theorem.

Theorem 2.11 (Banach-Mazur, Oxtoby)**.** *Let X be a nonempty topological space and let $A \subseteq X$. Then A is comeagre $\iff$ II has a winning strategy in $G^{\star\star}(A)$.*

In order to prove it we add an auxilary definition and lemma.

Definition 2.12. Let $S \subseteq \sigma$ be a pruned subtree of tree of all legal positions T and let $p = (U_0, V_0, \ldots, V_n) \in S$. We say that S is *comprehensive for p* if the family $\mathcal{V}_p = \{V_{n+1} \mid (U_0, V_0, \ldots, V_n, U_{n+1}, V_{n+1}) \in S\}$ (it may be that $n = -1$, which means $p = \emptyset$) is pairwise disjoint and $\bigcup \mathcal{V}_p$ is dense in V_n (where we think that $V_{-1} = X$). We say that S is *comprehensive* if it is comprehensive for each $p = (U_0, V_0, \ldots, V_n) \in S$.

Fact 2.13. *If σ is a winnig strategy for II then there exists a nonempty comprehensive $S \subseteq \sigma$.*

Proof. We construct S recursively as follows:

(1) $\emptyset \in S$,
(2) if $(U_0, V_0, \ldots, U_n) \in S$, then $(U_0, V_0, \ldots, U_n, V_n) \in S$ for the unique V_n given by the strategy σ,
(3) let $p = (U_0, V_0, \ldots, V_n) \in S$. For a possible player I's move $U_{n+1} \subseteq V_n$ let $U^\star_{n+1}$ be the unique set player II would respond with by σ. Now, by Zorn's Lemma, let $\mathcal{U}_p$ be a maximal collection of nonempty open subsets $U_{n+1} \subseteq V_n$ such that the set $\{U^\star_{n+1} \mid U_{n+1} \in \mathcal{U}_p\}$ is pairwise disjoint. Then put in S all $(U_0, V_0, \ldots, V_n, U_{n+1})$ such that $U_{n+1} \in \mathcal{U}_p$. This way S is comprehensive for p: the family $\mathcal{V}_p = \{V_{n+1} \mid (U_0, V_0, \ldots, V_n, U_{n+1}, V_{n+1}) \in S\}$ is exactly $\{U^\star_{n+1} \mid U_{n+1} \in \mathcal{U}_p\}$, which is pairwise disjoint and $\bigcup \mathcal{V}_p$ is obviously dense in V_n by the maximality of $\mathcal{U}_p$ – if there was any open set $\tilde{U}_{n+1} \subseteq V_n$ disjoint from $\bigcup \mathcal{V}_p$, then $\tilde{U}^\star_{n+1} \subseteq \tilde{U}_{n+1}$ would be also disjoint from $\bigcup \mathcal{V}_p$, so the family $\mathcal{U}_p \cup \{\tilde{U}_{n+1}\}$ would violate the maximality of $\mathcal{U}_p$. □

Lemma 2.14. *Let S be a nonempty comprehensive pruned subtree of a strategy σ. Then:*

(i) For any open $V_n \subseteq X$ there is at most one $p = (U_0, V_0, \ldots, U_n, V_n) \in S$.
(ii) Let $S_n = \{V_n \mid (U_0, V_0, \ldots, V_n) \in S\}$ for $n \in \mathbb{N}$ (i.e. S_n is a family of all possible choices player II can make in its n-th move according to S). Then $\bigcup S_n$ is open and dense in X.
(iii) S_n is a family of pairwise disjoint sets.

Proof. (i): Suppose that there are some $p = (U_0, V_0, \ldots, U_n, V_n)$, $p' = (U'_0, V'_0, \ldots, U'_n, V'_n)$ such that $V_n = V'_n$ and $p \neq p'$. Let k be the smallest index such that those sequences differ. We have two possibilities:

- $U_k = U'_k$ and $V_k \neq V'_k$ – this cannot be true simply by the fact that S is a subset of a strategy (so V_k is unique for U_k).
- $U_k \neq U'_k$: by the comprehensiveness of S we know that for $q = (U_0, V_0, \ldots, U_{k-1}, V_{k-1})$ the set $\mathscr{V}_q$ is pairwise disjoint. Thus $V_k \cap V'_k = \emptyset$, because $V_k, V'_k \in \mathscr{V}_q$. But this leads to a contradiction – V_n cannot be a nonempty subset of both V_k, V'_k.

(ii): The lemma is proved by induction on n. For $n = 0$ it follows trivially from the definition of comprehensiveness. Now suppose the lemma is true for n. Then the set $\bigcup_{V_n \in S_n} \bigcup \mathscr{V}_{p_{V_n}}$ (where p_{V_n} is given uniquely from (i)) is dense and open in X by the induction hypothesis. But $\bigcup S_{n+1}$ is exactly this set, thus it is dense and open in X.

(iii): We will prove it by induction on n. Once again, the case $n = 0$ follows from the comprehensiveness of S. Now suppose that the sets in S_n are pairwise disjoint. Take some $x \in V_{n+1} \in S_{n+1}$. Of course $\bigcup S_n \supseteq \bigcup S_{n+1}$, thus by the inductive hypothesis $x \in V_n$ for the unique $V_n \in S_n$. It must be that $V_{n+1} \in \mathscr{V}_{p_{V_n}}$, because V_n is the only superset of V_{n+1} in S_n. But $\mathscr{V}_{p_{V_n}}$ is disjoint, so there is no other $V'_{n+1} \in \mathscr{V}_{p_{V_n}}$ suc h that $x \in V'_{n+1}$. Moreover, there is no such set in $S_{n+1} \setminus \mathscr{V}_{p_{V_n}}$, because those sets are disjoint from V_n. Hence there is no $V'_{n+1} \in S_{n+1}$ other than V_n such that $x \in V'_{n+1}$. We've chosen x and V_{n+1} arbitrarily, so S_{n+1} is pairwise disjoint. □

Now we can move to the proof of the Banach-Mazur theorem.

Proof of theorem 2.11. ⇒: Let (A_n) be a sequence of dense open sets with $\bigcap_n A_n \subseteq A$. The simply *II* plays $V_n = U_n \cap A_n$, which is nonempty by the denseness of A_n.

⇐: Suppose *II* has a winning strategy σ. We will show that A is comeagre. Take a comprehensive $S \subseteq \sigma$. We claim that $\mathscr{S} = \bigcap_n \bigcup S_n \subseteq A$. By the lemma 2.14, (ii) sets $\bigcup S_n$ are open and dense, thus A must be comeagre. Now we prove the claim towards contradiction.

Suppose there is $x \in \mathscr{S} \setminus A$. By the lemma 2.14, (iii) for any n there is unique $x \in V_n \in S_n$. It follows that $p_{V_0} \subset p_{V_1} \subset \ldots$. Now the game $(U_0, V_0, U_1, V_1, \ldots) = \bigcup_n p_{V_n} \in [S] \subseteq [\sigma]$ is not winning for player *II*, which contradicts the assumption that σ is a winning strategy. □

Corollary 2.15. *If we add a constraint to the Banach-Mazur game such that players can only choose basic open sets, then the theorem 2.11 still suffices.*

Proof. If one adds the word *basic* before each occurrence of word *open* in previous proofs and theorems then they all will still be valid (except for ⇒, but its an easy fix – take V_n a basic open subset of $U_n \cap A_n$). □

This corollary will be important in using the theorem in practice – it's much easier to work with basic open sets rather than any open sets.

3. Fraïssé classes

In this section we will take a closer look at classes of finitely generated structures with some characteristic properties. More specifically, we will describe a concept developed by a French mathematician Roland Fraïssé called Fraïssé limit.

3.1. **Definitions.**

Definition 3.1. Let L be a signature and M be an L-structure. The *age* of M is the class $\mathbb{K}$ of all finitely generated structures that embedds into M. The age of M is also associated with class of all structures embeddable in M *up to isomorphism*.

Definition 3.2. We say that M has *countable age* when its age has countably many isomorphism types of finitely generated structures.

Definition 3.3. Let $\mathbb{K}$ be a class of finitely generated structures. $\mathbb{K}$ has *hereditary property (HP)* if for any $A \in \mathbb{K}$, any finitely generated substructure B of A it holds that $B \in \mathbb{K}$.

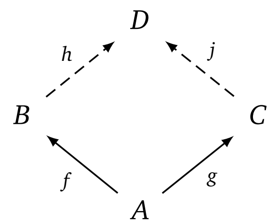
Definition 3.4. Let $\mathbb{K}$ be a class of finitely generated structures. We say that $\mathbb{K}$ has *joint embedding property (JEP)* if for any $A, B \in \mathbb{K}$ there is a structure $C \in \mathbb{K}$ such that both A and B embed in C.

Fraïssé has shown fundamental theories regarding age of a structure, one of them being the following one:

Fact 3.5. *Suppose L is a signature and $\mathbb{K}$ is a nonempty finite or countable set of finitely generated L-structures. Then $\mathbb{K}$ has the HP and JEP if and only if $\mathbb{K}$ is the age of some finite or countable structure.*

Beside the HP and JEP Fraïssé has distinguished one more property of the class $\mathbb{K}$, namely amalgamation property.

Definition 3.6. Let $\mathbb{K}$ be a class of finitely generated L-structures. We say that $\mathbb{K}$ has the *amalgamation property (AP)* if for any $A, B, C \in \mathbb{K}$ and embeddings $f : A \to B, g : A \to C$ there exists $D \in \mathbb{K}$ together with embeddings $h : B \to D$ and $j : C \to D$ such that $h \circ f = j \circ g$.



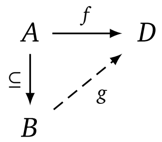
Definition 3.7. Let M be an L-structure. M is *ultrahomogenous* if every isomorphism between finitely generated substrucutres of M extends to an automorphism of M.

Having those definitions we can provide the main Fraïssé theorem.

Theorem 3.8 (Fraïssé theorem)**.** *Let L be a countable language and let $\mathbb{K}$ be a nonempty countable set of finitely generated L-structures which has HP, JEP and AP. Then $\mathbb{K}$ is the age of a countable, ultrahomogenous L-structure M. Moreover, M is unique up to isomorphism. We say that M is a* Fraïssé limit *of $\mathbb{K}$ and denote this by $M = \mathrm{FLim}(\mathbb{K})$.*

This is a well known theorem. One can read a proof of this theorem in Wilfrid Hodges' classical book *Model Theory* [1]. In the proof of this theorem appears another, equally important 3.10.

Definition 3.9. We say that an L-structure M is *weakly ultrahomogenous* if for any A, B finitely generated substructures of M such that $A \subseteq B$ and an embedding $f : A \to M$ there is an embedding $g : B \to M$ which extends f.



Lemma 3.10. *A countable structure is ultrahomogenous if and only if it is weakly ultrahomogenous.*

This lemma will play a major role in the later parts of the paper. Weak ultrahomogenity is an easier and more intuitive property and it will prove useful when recursively constructing the generic automorphism of a Fraïssé limit.

References

[1] Wilfrid Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993. DOI: 10.1017/CBO9780511551574.