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Generic automorphisms as Fraïssé limits

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1. Introduction

Model theory is a field of mathematics that classifies and constructs structures with particular properties (particularly those expressible in first order logic). It describes classical mathematical objects in a broader context, abstracts their properties and studies connections between seemingly unrelated structures. This work studies limits of Fraïssé classes with additional combinatorial and categorical properties. Fraïssé classes are frequently used in model theory, both as a source of examples and to analyse particular "generic" structures.

The notion of Fraïssé class and its limit is due to the French logician Roland Fraïssé. He also introduced the back-and-forth argument, a fundamental model theoretical method in construction of elementarily equivalent structures, upon which Ehrenfeucht-Fraïssé games are based.

The prototypical example for this paper is the random graph 3.13 (also known as the Rado graph), the Fraïssé limit of the class of finite undirected graphs. It serves as a useful example, gives an intuition of the Fraïssé limits, weak Hrushovski property and free amalgamation. Perhaps most importantly, the random graph has a so-called generic automorphism 2.6, which was first proved by Truss in [9], where he also introduced the term.

The key Theorem 4.5 says that a Fraïssé class with canonical amalgamation and weak Hrushovski property has a generic automorphism. The fact that such an automorphism exists in this case follows from the classical results of Ivanov [3] and Kechris-Rosendal [5], we show a new way to construct a generic automorphism by expanding the structures of the class by a (total) automorphism and considering limit of such extended Fraïssé class. We achieve this by using the Banach-Mazur games, a well known method in the descriptive set theory, which proves useful in the study of comeagre sets.

Finally, we show how this construction of the generic automorphism can be used to deduce some properties of generic automorphisms (see 4.6, (COŚ JESZCZE)).

Wstęp

Teoria modeli jest działem matematyki zajmującym się klasyfikacją i konstrukcją struktur z określonymi cechami (szczególnie takimi, które da się wyrazić logiką pierwszego rzędu). Opisuje klasyczne matematyczne obiekty w szerszym kontekście, abstrahuje ich własności i opisuje połączenia między pozornie niepowiązanymi strukturami. Niniejsza praca bada granicę klas Fraïsségo z dodatkowymi kombinatorycznymi i kategoryjnymi własnościami. Klasy Fraïsségo są powszechnie znanym i używanym konceptem w teorii modeli, zarówno jako narzędzie opisujące "generyczne" struktury, jak i źródło przykładów.

Klasy Fraïsségo i ich granice zostały opisane po raz pierwszy przez francuskiego logika Rolanda Fraïsségo. Zawdzięczamy mu również argument "back-and-forth", fundamentalną teoriomodelową metodę konstrukcji elementarnie równoważnych struktur, na podstawie której bazują gry Ehrenfeuchta-Fraïsségo.

Graf losowy 3.13, zwany również grafem Rado, jest prototypową strukturą tej prac. Graf losowy można skonstruować jako granicę Fraïsségo klasy skończonych grafów nieskierowanych. Służy on jako użyteczny przykład, daje intuicję stojącą za konstrukcją granicy Fraïsségo, słabej własności Hrushovskiego oraz wolnej amalgamacji. Ponadto, co najważniejsze dla niniejszej pracy, graf losowy ma tak zwany *generyczny automorfizm* 2.6, co zostało po raz pierwsze zdefiniowane i udowodnione przez Trussa w [9].

Kluczowe twierdzenie 4.5 mówi, że klasa Fraïsségo z kanoniczną amalgamacją i słabą własnością Hrushovskiego ma generyczny automorfizm. Istnienie takiego automorfizmu w tym przypadku wynika z wcześniejszych klasycznych wyników Ivanova [3] oraz Kechrisa-Rosendala [5]. W tej pracy pokazujemy nowy sposób konstrukcji generycznego automorfizmu poprzez rozszerzenie struktur klasy o (totalny) automorfizm oraz analizę granicy Fraïsségo nowo powstałej klasy. Posługujemy się przy tym grami Banacha-Mazura, które są dobrze znanym narzędziem w deskryptywnej teorii mnogości.

Opisana konstrukcja generycznego automorfizmu okazuje się pomocna w dowodzeniu niektórych własności tego automorfizmu (patrz 4.6).

2. Preliminaries

Before we get to the main work of the paper, we need to establish basic notions, known facts and theorems. This section provides a brief introduction to the theory of Baire spaces and category theory. Most of the notions are well known, interested reader may look at [4], [6]

2.1. **Descriptive set theory.** In this section we provide an important definition of a *comeagre* set. It is purely topological notion, the intuition may come from the measure theory though. For example, in a standard Lebesuge measure on the real interval $[0,1]$, the set of rationals is of measure 0, although being a dense subset of the $[0,1]$. So, in a sense, the set of rationals is *meagre* in the interval $[0,1]$. On the other hand, the set of irrational numbers is also dense, but have measure 1, so it is *comeagre*.

This is only a rough approximation of the topological definition. The definitions are based on the Kechris' book *Classical Descriptive Set Theory* [4]. One should look into it for more details and examples.

Definition 2.1. Suppose X is a topological space and $A \subseteq X$. We say that A is *meagre* in X if $A = \bigcup_{n\in\mathbb{N}} A_n$, where A_n are nowhere dense subsets of X (i.e. $\mathrm{Int}(\bar{A_n}) = \emptyset$).

Definition 2.2. We say that A is *comeagre* in X if it is a complement of a meagre set. Equivalently, a set is comeagre if and only if it contains a countable intersection of open dense sets.

Every countable set is meagre in any T_1 space. So, $\mathbb{Q}$ is meagre in $\mathbb{R}$ (although it is dense), which means that the set of irrationals is comeagre. The Cantor set is nowhere dense, hence meagre in the $[0,1]$ interval.

Definition 2.3. We say that a topological space X is a *Baire space* if every comeagre subset of X is dense in X (equivalently, every meagre set has empty interior).

Definition 2.4. Suppose X is a Baire space. We say that a property P *holds generically* for a point $x \in X$ if $\{x \in X \mid P \text{ holds for } x\}$ is comeagre in X.

Let M be a structure. We define a topology on the automorphism group $\mathrm{Aut}(M)$ of M by the basis of open sets: for a finite function $f : M \to M$ we have a basic open set $[f]_{\mathrm{Aut}(M)} = \{g \in \mathrm{Aut}(M) \mid f \subseteq g\}$. This is a standard definition.

Fact 2.5. *For a countable structure M, the topological space* $\mathrm{Aut}(M)$ *is a Baire space.*

This is in fact a very weak statement, it is also true that $\mathrm{Aut}(M)$ is a Polish space (i.e. separable completely metrizable), and every Polish space is Baire. However, those additional properties are not important in this study.

Definition 2.6. Let $G = \mathrm{Aut}(M)$ be the automorphism group of structure M. We say that $f \in G$ is a *generic automorphism*, if the conjugacy class of f is comeagre in G.

Definition 2.7. Let X be a nonempty topological space and let $A \subseteq X$. The *Banach-Mazur game of A*, denoted as $G^{\star\star}(A)$ is defined as follows: Players I

and *II* take turns in playing nonempty open sets $U_0, V_0, U_1, V_1, \dots$ such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$. We say that player *II* wins the game if $\bigcap_n V_n \subseteq A$.

There is an important Theorem 2.13 on the Banach-Mazur game: A is comeagre if and only if *II* can always choose sets $V_0, V_1, \dots$ such that it wins. Before we prove it we need to define notions necessary to formalise and prove the theorem.

Definition 2.8. T is *the tree of all legal positions* in the Banach-Mazur game $G^{\star\star}(A)$ when T consists of all finite sequences $(W_0, W_1, \dots, W_n)$, where W_i are nonempty open sets such that $W_0 \supseteq W_1 \supseteq \dots \supseteq W_n$.

Definition 2.9. We say that σ is *a pruned subtree* of the tree of all legal positions T if $\sigma \subseteq T$, for any $(W_0, W_1, \dots, W_n) \in \sigma, n \geq 0$ there is a W such that $(W_0, W_1, \dots, W_n, W) \in \sigma$ (it simply means that there's no finite branch in σ) and $(W_0, W_1, \dots W_{n-1}) \in \sigma$ (every node on a branch is in σ).

Definition 2.10. Let σ be a pruned subtree of the tree of all legal positions T. By $[\sigma]$ we denote *the set of all infinite branches of* σ, i.e. infinite sequences $(W_0, W_1, \dots)$ such that $(W_0, W_1, \dots W_n) \in \sigma$ for any $n \in \mathbb{N}$.

Definition 2.11. A *strategy* for *II* in $G^{\star\star}(A)$ is a pruned subtree $\sigma \subseteq T$ such that

(i) σ is nonempty,
(ii) if $(U_0, V_0, \dots, U_n, V_n) \in \sigma$, then for all open nonempty $U_{n+1} \subseteq V_n$, $(U_0, V_0, \dots, U_n, V_n, U_{n+1}) \in \sigma$,
(iii) if $(U_0, V_0, \dots, U_n) \in \sigma$, then for a unique V_n, $(U_0, V_0, \dots, U_n, V_n) \in \sigma$.

Intuitively, a strategy σ works as follows: I starts playing U_0 as any open subset of X, then *II* plays unique (by (iii)) V_0 such that $(U_0, V_0) \in \sigma$. Then I responds by playing any $U_1 \subseteq V_0$ and *II* plays unique V_1 such that $(U_0, V_0, U_1, V_1) \in \sigma$, etc.

We will often denote a sequence $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$ of open sets as *an instance* of a Banach-Mazur game, or just simply by a *game*.

Definition 2.12. A strategy σ is a *winning strategy for II* if for any instance $(U_0, V_0 \dots) \in [\sigma]$ of the Banach-Mazur game player *II* wins, i.e. $\bigcap_n V_n \subseteq A$.

Theorem 2.13 (Banach-Mazur, Oxtoby)**.** *Let X be a nonempty topological space and let $A \subseteq X$. Then A is comeagre $\Longleftrightarrow$ II has a winning strategy in $G^{\star\star}(A)$.*

The statement of the theorem is once again taken from Kechris [4] 8.33. However, the proof given in the book is brief, thus we present a detailed version. In order to prove the theorem we add an auxiliary definition and lemma.

Definition 2.14. Let $S \subseteq \sigma$ be a pruned subtree of tree of all legal positions T and let $p = (U_0, V_0, \dots, V_n) \in S$. We say that S is *comprehensive for p* if the family $\mathcal{V}_p = \{V_{n+1} \mid (U_0, V_0, \dots, V_n, U_{n+1}, V_{n+1}) \in S\}$ (it may be that $n = -1$, which means $p = \emptyset$) is pairwise disjoint and $\bigcup \mathcal{V}_p$ is dense in V_n (where we put $V_{-1} = X$). We say that S is *comprehensive* if it is comprehensive for each $p = (U_0, V_0, \dots, V_n) \in S$.

Fact 2.15. *If σ is a winning strategy for II then there exists a nonempty comprehensive $S \subseteq \sigma$.*

Proof. We construct S recursively as follows:

(1) $\emptyset \in S$,
(2) if $(U_0, V_0, \dots, U_n) \in S$, then $(U_0, V_0, \dots, U_n, V_n) \in S$ for the unique V_n given by the strategy σ,
(3) let $p = (U_0, V_0, \dots, V_n) \in S$. For a possible player move of player I $U_{n+1} \subseteq V_n$ let $U^\star_{n+1}$ be the unique set player *II* would respond with by σ. Now, by Zorn's Lemma, let $\mathscr{U}_p$ be a maximal collection of nonempty open subsets $U_{n+1} \subseteq V_n$ such that the set $\{U^\star_{n+1} \mid U_{n+1} \in \mathscr{U}_p\}$ is pairwise disjoint. Then put in S all $(U_0, V_0, \dots, V_n, U_{n+1})$ such that $U_{n+1} \in \mathscr{U}_p$. This way S is comprehensive for p: the family $\mathscr{V}_p = \{V_{n+1} \mid (U_0, V_0, \dots, V_n, U_{n+1}, V_{n+1}) \in S\}$ is exactly $\{U^\star_{n+1} \mid U_{n+1} \in \mathscr{U}_p\}$, which is pairwise disjoint and $\bigcup \mathscr{V}_p$ is obviously dense in V_n by the maximality of $\mathscr{U}_p$ – if there was any open set $\tilde{U}_{n+1} \subseteq V_n$ disjoint from $\bigcup \mathscr{V}_p$, then $\tilde{U}^\star_{n+1} \subseteq \tilde{U}_{n+1}$ would be also disjoint from $\bigcup \mathscr{V}_p$, so the family $\mathscr{U}_p \cup \{\tilde{U}_{n+1}\}$ would violate the maximality of $\mathscr{U}_p$. □

Lemma 2.16. *Let S be a nonempty comprehensive pruned subtree of a strategy σ. Then:*

(i) For any open $V_n \subseteq X$ there is at most one $p = (U_0, V_0, \dots, U_n, V_n) \in S$.

Let $S_n = \{V_n \mid (U_0, V_0, \dots, V_n) \in S\}$ for $n \in \mathbb{N}$ (i.e. S_n is a family of all possible choices player II can make in its n-th move according to S).

(ii) $\bigcup S_n$ is open and dense in X.
(iii) S_n is a family of pairwise disjoint sets.

Proof. (i): Suppose that there are some $p = (U_0, V_0, \dots, U_n, V_n)$, $p' = (U'_0, V'_0, \dots, U'_n, V'_n)$ such that $V_n = V'_n$ and $p \neq p'$. Let k be the smallest index such that those sequences differ. We have two possibilities:

- $U_k = U'_k$ and $V_k \neq V'_k$ – this cannot be true simply by the fact that S is a subset of a strategy (so V_k is unique for U_k).
- $U_k \neq U'_k$: by the comprehensiveness of S we know that for $q = (U_0, V_0, \dots, U_{k-1}, V_{k-1})$ the set $\mathscr{V}_q$ is pairwise disjoint. Thus $V_k \cap V'_k = \emptyset$, because $V_k, V'_k \in \mathscr{V}_q$. But this leads to a contradiction – V_n cannot be a nonempty subset of both V_k, V'_k.

(ii): The lemma is proved by induction on n. For $n = 0$ it follows trivially from the definition of comprehensiveness. Now suppose the lemma is true for n. Then the set $\bigcup_{V_n \in S_n} \bigcup \mathscr{V}_{p_{V_n}}$ (where p_{V_n} is given uniquely from (i)) is dense and open in X by the induction hypothesis. But $\bigcup S_{n+1}$ is exactly this set, thus it is dense and open in X.

(iii): We will prove it by induction on n. Once again, the case $n = 0$ follows from the comprehensiveness of S. Now suppose that the sets in S_n are pairwise disjoint. Take some $x \in V_{n+1} \in S_{n+1}$. Of course $\bigcup S_n \supseteq \bigcup S_{n+1}$, thus by the inductive hypothesis $x \in V_n$ for the unique $V_n \in S_n$. It must be that $V_{n+1} \in \mathscr{V}_{p_{V_n}}$, because V_n is the only superset of V_{n+1} in S_n. But $\mathscr{V}_{p_{V_n}}$ is disjoint, so there is no other $V'_{n+1} \in \mathscr{V}_{p_{V_n}}$ such that $x \in V'_{n+1}$. Moreover, there is no such set in $S_{n+1} \setminus \mathscr{V}_{p_{V_n}}$, because those sets are disjoint from V_n. Hence there is no $V'_{n+1} \in S_{n+1}$ other than V_n such that $x \in V'_{n+1}$. We have chosen x and V_{n+1} arbitrarily, so S_{n+1} is pairwise disjoint. □

Now we can move to the proof of the Banach-Mazur theorem.

Proof of Theorem 2.13. $\Rightarrow$: Let (A_n) be a sequence of dense open sets with $\bigcap_n A_n \subseteq A$. The simply *II* plays $V_n = U_n \cap A_n$, which is nonempty by the denseness of A_n.

$\Leftarrow$: Suppose *II* has a winning strategy σ. We will show that A is comeagre. Take a comprehensive $S \subseteq \sigma$. We claim that $\mathscr{S} = \bigcap_n \bigcup S_n \subseteq A$. By the lemma 2.16, (ii) sets $\bigcup S_n$ are open and dense, thus A must be comeagre. Now we prove the claim towards contradiction.

Suppose there is $x \in \mathscr{S} \setminus A$. By the lemma 2.16, (iii) for any n there is unique $x \in V_n \in S_n$. It follows that $p_{V_0} \subset p_{V_1} \subset \ldots$. Now the game $(U_0, V_0, U_1, V_1, \ldots) = \bigcup_n p_{V_n} \in [S] \subseteq [\sigma]$ is not winning for player *II*, which contradicts the assumption that σ is a winning strategy. □

Corollary 2.17. *If we add a constraint to the Banach-Mazur game such that players can only choose basic open sets, then the Theorem 2.13 still suffices.*

Proof. If one adds the word *basic* before each occurrence of word *open* in previous proofs and theorems then they all will still be valid (except for $\Rightarrow$, but its an easy fix – take for V_n a basic open subset of $U_n \cap A_n$). □

This corollary will be important in using the theorem in practice – it's much easier to work with basic open sets rather than arbitrary open sets.

2.2. **Category theory.** In this section we will give a short introduction to the notions of category theory that will be necessary to generalize the key result of the paper.

We will use a standard notation. If the reader is interested in a more detailed introduction to the category theory, then it's recommended to take a look at [6]. Here we will shortly describe the standard notation.

A *category* $\mathscr{C}$ consists of the collection of objects (denoted as $\mathrm{Obj}(\mathscr{C})$, but most often simply as $\mathscr{C}$) and collection of *morphisms* $\mathrm{Mor}(A,B)$ between each pair of objects $A, B \in \mathscr{C}$. We require that for each pair of morphisms $f : B \to C$, $g : A \to B$ there is a morphism $f \circ g : A \to C$. If $f : A \to B$ then we say that A is the domain of f $(\mathrm{Dom}\, f)$ and that B is the range of f $(\mathrm{Rng}\, f)$.

For every $A \in \mathscr{C}$ there is an *identity morphism* id_A such that for any morphism $f \in \mathrm{Mor}(A,B)$ we have that $f \circ id_A = \mathrm{id}_B \circ f$.

We say that $f : A \to B$ is *isomorphism* if there is (necessarily unique) morphism $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. Automorphism is an isomorphism where $A = B$.

A *functor* is a "(homo)morphism" of categories. We say that $F : \mathscr{C} \to \mathscr{D}$ is a functor from category $\mathscr{C}$ to category $\mathscr{D}$ if it associates each object $A \in \mathscr{C}$ with an object $F(A) \in \mathscr{D}$, associates each morphism $f : A \to B$ in $\mathscr{C}$ with a morphism $F(f) : F(A) \to F(B)$. We also require that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and that for any (compatible) morphisms f, g in $\mathscr{C}$ $F(f \circ g) = F(f) \circ F(g)$.

In category theory we distinguish *covariant* and *contravariant* functors. Here, we only consider covariant functors, so we will simply say *functor*.

Fact 2.18. *Functor $F : \mathscr{C} \to \mathscr{D}$ maps isomorphism $f : A \to B$ in $\mathscr{C}$ to the isomorphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$.*

A notion that will be very important for us is a "morphism of functors" which is called *natural transformation*.

Definition 2.19. Let F, G be functors between the categories $\mathscr{C}, \mathscr{D}$. A *natural transformation* τ is function that assigns to each object A of $\mathscr{C}$ a morphism τ_A in $\mathrm{Mor}(F(A), G(A))$ such that for every morphism $f : A \to B$ in $\mathscr{C}$ the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ B & F(B) \arrow[r, "\tau_B"] & G(B) \end{tikzcd}$$

Natural transformation has, *nomen omen*, natural properties. One particularly interesting to us is the following fact.

Fact 2.20. *Let η be a natural transformation of functors F, G from category $\mathscr{C}$ to $\mathscr{D}$. Then η is an isomorphism if and only if all of the component morphisms are isomorphisms.*

Proof. Suppose that η_A is an isomorphism for every $A \in \mathscr{C}$, where $\eta_A \colon F(A) \to G(A)$ is the morphism of the natural transformation corresponding to A. Then η^{-1} is simply given by the morphisms η_A^{-1}.

Now assume that η is an isomorphism, i.e. $\eta^{-1} \circ \eta = \mathrm{id}_F$. *Ad contrario* assume that there is $A \in \mathscr{C}$ such that the component morphism $\eta_A \colon F(A) \to G(A)$ is not an isomorphism. It means that $\eta_A^{-1} \circ \eta_A \neq id_A$, hence $F(A) = \mathrm{Dom}(\eta^{-1} \circ \eta)(A) \neq \mathrm{Rng}(\eta^{-1} \circ \eta)(A) = F(A)$, which is obviously a contradiction. □

Definition 2.21. In category theory, a *diagram* of type $\mathscr{J}$ in category $\mathscr{C}$ is a functor $D \colon \mathscr{J} \to \mathscr{C}$. $\mathscr{J}$ is called the *index category* of D. In other words, D is of *shape* $\mathscr{J}$.

For example, $\mathscr{J} = \{-1 \leftarrow 0 \rightarrow 1\}$, then a diagram $D \colon \mathscr{J} \to \mathscr{C}$ is called a *cospan*. For example, if A, B, C are objects of $\mathscr{C}$ and $f \in \mathrm{Mor}(C, A), g \in \mathrm{Mor}(C, B)$, then the following diagram is a cospan:

$$\begin{tikzcd} A & & B \\ & C \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

From now we omit explicit definition of the index category, as it is easily referable from a picture.

Definition 2.22. Let A, B, C, D be objects in the category $\mathscr{C}$ with morphisms $e \colon C \to A, f : C \to B, g : A \to D, h \colon B \to D$ such that $g \circ e = h \circ f$. Then the following diagram:

$$\begin{tikzcd} & D & \\ A \arrow[ur, "g"] & & B \arrow[ul, "h"'] \\ & C \arrow[ul, "f"] \arrow[ur, "e"'] & \end{tikzcd}$$

is called a *pushout diagram*.

In both definitions of cospan and pushout diagrams we say that the object C is the *base* of the diagram.

Definition 2.23. The *cospan category* of category $\mathscr{C}$, referred to as Cospan($\mathscr{C}$), is the category of cospan diagrams of $\mathscr{C}$, where morphisms between two cospans are natural transformations of the underlying functors.

We define *pushout category* analogously and call it Pushout($\mathscr{C}$).

From now on we work in subcategories of cospan diagrams and pushout diagrams where we fix the base structure. Formally, for a fixed $C \in \mathscr{C}$, category $\text{Cospan}_C(\mathscr{C})$ is the category of all cospans in Cospan($\mathscr{C}$) such that the base of the diagram is C. Natural transformation η of two diagrams in $\text{Cospan}_C(\mathscr{C})$ are such that the morphism $\eta_C : C \to C$ is an automorphism of C. $\text{Pushout}_C(\mathscr{C})$ is defined analogously. In most contexts we consider only one base structure, hence we will often write Pushout($\mathscr{C}$) instead of $\text{Pushout}_C(\mathscr{C})$.

3. Fraïssé classes

In this section we will take a closer look at classes of finitely generated structures with some characteristic properties. More specifically, we will describe a concept developed by a French mathematician Roland Fraïssé called Fraïssé limit.

3.1. **Definitions.**

Definition 3.1. Let L be a signature and M be an L-structure. The *age* of M is the class $\mathscr{K}$ of all finitely generated structures that embed into M. The age of M is also associated with class of all structures embeddable in M *up to isomorphism*.

Definition 3.2. We say that a class $\mathscr{K}$ of finitely generated structures is *essentially countable* if it has countably many isomorphism types of finitely generated structures.

Definition 3.3. Let $\mathscr{K}$ be a class of finitely generated structures. $\mathscr{K}$ has the *hereditary property (HP)* if for any $A \in \mathscr{K}$ and any finitely generated substructure B of A it holds that $B \in \mathscr{K}$.

Definition 3.4. Let $\mathscr{K}$ be a class of finitely generated structures. We say that $\mathscr{K}$ has the *joint embedding property (JEP)* if for any $A, B \in \mathscr{K}$ there is a structure $C \in \mathscr{K}$ such that both A and B embed in C.

$$\begin{tikzcd} & C & \\ A \arrow[ur, dashed, "f"] & & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

In terms of category theory we may say that $\mathscr{K}$ is a category of finitely generated structures where morphisms are embeddings of those structures. Then the above diagram is a *span* diagram in category $\mathscr{K}$.

Fraïssé has shown fundamental theorems regarding age of a structure, one of them being the following one:

Fact 3.5. *Suppose L is a signature and $\mathscr{K}$ is a nonempty essentially countable set of finitely generated L-structures. Then $\mathscr{K}$ has the HP and JEP if and only if $\mathscr{K}$ is the age of some finite or countable structure.*

Proof. One can read a proof of this fact in Wilfrid Hodges' classical book *Model Theory* [1, Theorem 7.1.1]. □

Beside the HP and JEP Fraïssé has distinguished one more property of the class $\mathscr{K}$, namely the amalgamation property.

Definition 3.6. Let $\mathscr{K}$ be a class of finitely generated L-structures. We say that $\mathscr{K}$ has the *amalgamation property (AP)* if for any $A, B, C \in \mathscr{K}$ and embeddings $e : C \to A, f : C \to B$ there exists $D \in \mathscr{K}$ together with embeddings $g : A \to D$ and $h : A \to D$ such that $g \circ e = h \circ f$.

$$\begin{tikzcd}
& D & \\
A \arrow[ur, dashed, "g"] & & B \arrow[ul, dashed, "h"'] \\
& C \arrow[ul, "e"] \arrow[ur, "f"'] &
\end{tikzcd}$$

In terms of category theory, $\mathscr{K}$ has the amalgamation property if every cospan diagram can be extended to a pushout diagram in category $\mathscr{K}$. We will get into more details later, in the definition of canonical amalgamation 3.19.

Definition 3.7. Class $\mathscr{K}$ of finitely generated structure is a *Fraïssé class* if it is essentially countable, has HP, JEP and AP.

Definition 3.8. Let M be an L-structure. M is *ultrahomogeneous* if every isomorphism between finitely generated substructures of M extends to an automorphism of M.

Having those definitions we can provide the main Fraïssé theorem.

Theorem 3.9 (Fraïssé theorem)**.** *Let L be a countable language and let $\mathscr{K}$ be a nonempty countable set of finitely generated L-structures which has HP, JEP and AP. Then $\mathscr{K}$ is the age of a countable, ultrahomogeneous L-structure M. Moreover, M is unique up to isomorphism. We say that M is a* Fraïssé limit *of $\mathscr{K}$ and denote this by $M = \mathrm{Flim}(\mathscr{K})$.*

Proof. Check the proof in [1, theorem 7.1.2]. □

Definition 3.10. We say that an L-structure M is *weakly ultrahomogeneous* if for any A, B, finitely generated substructures of M, such that $A \subseteq B$ and an embedding $f : A \to M$ there is an embedding $g : B \to M$ which extends f.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\subseteq"'] & D \\ B \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Lemma 3.11. *A countable structure is ultrahomogeneous if and only if it is weakly ultrahomogeneous.*

Proof. Proof can be again found in [1, lemma 7.1.4(b)]. □

This lemma will play a major role in the later parts of the paper. Weak ultrahomogeneity is an easier and more intuitive property and it will prove useful when recursively constructing the generic automorphism of a Fraïssé limit.

3.2. **Random graph.** In this section we'll take a closer look on a class of finite undirected graphs, which is a Fraïssé class.

The language of undirected graphs L consists of a single binary relational symbol E. If G is an L-structure then we call it a *graph*, and its elements *vertices*. If for some vertices $u, v \in G$ we have $G \models uEv$ then we say that there is an *edge* connecting u and v. If $G \models \forall x \forall y (xEy \longleftrightarrow yEx)$ then we say that G is an *undirected graph*. From now on we omit the word *undirected* and consider only undirected graphs.

Proposition 3.12. *Let $\mathscr{G}$ be the class of all finite graphs. $\mathscr{G}$ is a Fraïssé class.*

Proof. $\mathscr{G}$ is of course countable (up to isomorphism) and has the HP (graph substructure is also a graph). It has JEP: having two finite graphs G_1, G_2 take their disjoint union $G_1 \sqcup G_2$ as the extension of them both. $\mathscr{G}$ has the AP. Having graphs A, B, C, where B and C are supergraphs of A, we can assume without loss of generality that $(B \setminus A) \cap (C \setminus A) = \emptyset$. Then $A \sqcup (B \setminus A) \sqcup (C \setminus A)$ is the graph we are looking for (with edges as in B and C and without any edges between $B \setminus A$ and $C \setminus A$). □

Definition 3.13. The *random graph* is the Fraïssé limit of the class of finite graphs $\mathscr{G}$ denoted by $\mathit{\Gamma} = \mathrm{Flim}(\mathscr{G})$.

The concept of the random graph emerges independently in many fields of mathematics. For example, one can construct the graph by choosing at random for each pair of vertices if they should be connected or not. It turns out that the graph constructed this way is isomorphic to the random graph with probability 1.

The random graph $\mathit{\Gamma}$ has one particular property that is unique to the random graph.

Fact 3.14 (random graph property)**.** *For each finite disjoint $X, Y \subseteq \mathit{\Gamma}$ there exists $v \in \mathit{\Gamma} \setminus (X \cup Y)$ such that $\forall u \in X$ we have that $\mathit{\Gamma} \models vEu$ and $\forall u \in Y$ We have that $\mathit{\Gamma} \models \neg vEu$.*

Proof. Take any finite disjoint $X, Y \subseteq \mathit{\Gamma}$. Let G_{XY} be the subgraph of $\mathit{\Gamma}$ induced by the $X \cup Y$. Let $H = G_{XY} \cup \{w\}$, where w is a new vertex that does not appear in G_{XY}. Also, w is connected to all vertices of G_{XY} that come from X and to none of those that come from Y. This graph is of course finite, so it is embeddable in $\mathit{\Gamma}$ by some $h \colon H \to \mathit{\Gamma}$. Let f be the partial isomorphism from

$X \sqcup Y$ to $h[H] \subseteq \Gamma$, with X and Y projected to the part of $h[H]$ that come from X and Y respectively. By the ultrahomogeneity of Γ this isomorphism extends to an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$. Then $v = \sigma^{-1}(w)$ is the vertex we sought. □

Fact 3.15. *If a countable graph G has the random graph property, then it is isomorphic to the random graph Γ.*

Proof. Enumerate vertices of both graphs: $\Gamma = \{a_1, a_2 \ldots\}$ and $G = \{b_1, b_2 \ldots\}$. We will construct a chain of partial isomorphisms $f_n \colon \Gamma \to G$ such that $\emptyset = f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ and $a_n \in \mathrm{Dom}(f_n)$ and $b_n \in \mathrm{Rng}(f_n)$ for each $n \in \mathbb{N}$.

Suppose we have f_n. We seek $b \in G$ such that $f_n \cup \{\langle a_{n+1}, b\rangle\}$ is a partial isomorphism. If $a_{n+1} \in \mathrm{Dom}\, f_n$, then simply $b = f_n(a_{n+1})$. Otherwise, let $X = \{a \in \Gamma \mid aE_\Gamma a_{n+1}\} \cap \mathrm{Dom}\, f_n, Y = X^c \cap \mathrm{Dom}\, f_n$, i.e. X are vertices of $\mathrm{Dom}\, f_n$ that are connected with a_{n+1} in Γ and Y are those vertices that are not connected with a_{n+1}. Let b be a vertex of G that is connected to all vertices of $f_n[X]$ and to none $f_n[Y]$ (it exists by the random graph property). Then $f_n \cup \{\langle a_{n+1}, b\rangle\}$ is a partial isomorphism. We find a for the b_{n+1} in the similar manner, so that $f_{n+1} = f_n \cup \{\langle a_{n+1}, b\rangle, \langle a, b_{n+1}\rangle\}$ is a partial isomorphism.

Finally, $f = \bigcup_{n=0}^{\infty} f_n$ is an isomorphism between Γ and G. Take any $a, b \in \Gamma$. Then for some big enough n we have that $aE_\Gamma b \iff f_n(a)E_G f_n(b) \iff f(a)E_G f(b)$. □

Using this fact one can show that the graph constructed in the probabilistic manner is in fact isomorphic to the random graph Γ.

Definition 3.16. We say that a Fraïssé class $\mathscr{K}$ has the *weak Hrushovski property* (*WHP*) if for every $A \in \mathscr{K}$ and an isomorphism of its finitely generated substructures $p : A \to A$ (also called a partial automorphism of A), there is some $B \in \mathscr{K}$ such that p can be extended to an automorphism of B, i.e. there is an embedding $i \colon A \to B$ and a $\bar{p} \in \mathrm{Aut}(B)$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, dashed, "\bar{p}"] & B \\ A \arrow[u, dashed, "i"] \arrow[r, "p"] & A \arrow[u, dashed, "i"'] \end{tikzcd}$$

Proposition 3.17. *The class of finite graphs $\mathscr{G}$ has the weak Hrushovski property.*

The proof of this proposition can be done directly, in a combinatorial manner, as shown in [7]. Hrushovski has also shown in [2] that finite graphs have stronger property, where each graph can be extended to a supergraph so that every partial automorphism of the graph extend to an automorphism of the supergraph.

Moreover, there is a theorem saying that every Fraïssé class $\mathscr{K}$, in a relational language L, with *free amalgamation* (see the definition 3.18 below) has WHP. The statement and proof of this theorem can be found in [8, theorem 3.2.8]. We provide the definition of free amalgamation that is coherent with the notions established in our paper.

Definition 3.18. Let L be a relational language and $\mathscr{K}$ a class of L-structures. $\mathscr{K}$ has *free amalgamation* if for every $A, B, C \in \mathscr{K}$ such that $C = A \cap B$ the following diagram commutes:

$$\begin{tikzcd}
 & A \sqcup_C B & \\
A \arrow[ur, hook] & & B \arrow[ul, hook'] \\
 & C \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

$A \sqcup_C B$ here is an L-structure with domain $A \cup B$ such that for every n-ary symbol R from L, n-tuple $\bar{a} \subseteq A \cup B$, we have that $A \sqcup B \models R(\bar{a})$ if and only if $[\bar{a} \subseteq A$ and $A \models R(\bar{a})]$ or $[\bar{a} \subseteq B$ and $B \models R(\bar{a})]$.

Actually we did already implicitly worked with free amalgamation in the Proposition 3.12, showing that the class of finite graphs is indeed a Fraïssé class.

3.3. **Canonical amalgamation.** Recall, Cospan($\mathscr{C}$), Pushout($\mathscr{C}$) are the categories of cospan and pushout diagrams of the category $\mathscr{C}$. We have also denoted the notion of cospans and pushouts with a fixed base structure C denoted as $\mathrm{Cospan}_C(\mathscr{C})$ and $Pushout_C(\mathscr{C})$.

Definition 3.19. Let $\mathscr{K}$ be a class finitely generated L-structures. We say that $\mathscr{K}$ has *canonical amalgamation* if for every $C \in \mathscr{K}$ there is a functor $\otimes_C \colon \mathrm{Cospan}_C(\mathscr{K}) \to \mathrm{Pushout}_C(\mathscr{K})$ such that it has the following properties:

- Let $A \leftarrow C \to B$ be a cospan. Then $\otimes_C$ sends it to a pushout that preserves "the bottom" structures and embeddings, i.e.:

$$\begin{tikzcd}
 & & & & A \otimes_C B & \\
A & & B \arrow[r, dashed, "A\otimes_C B"] & A \arrow[ur, dashed] & & B \arrow[ul, dashed] \\
 & C \arrow[ul] \arrow[ur] & & & C \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

  We have deliberately omitted names for embeddings of C. Of course, the functor has to take them into account, but this abuse of notation is convenient and should not lead into confusion.
- Let $A \leftarrow C \to B, A' \leftarrow C \to B'$ be cospans with a natural transformation η given by $\alpha \colon A \to A', \beta \colon B \to B', \gamma \colon C \to C$. Then $\otimes_C$ preserves the morphisms of η when sending it to the natural transformation of pushouts by adding the $\delta \colon A \otimes_C B \to A' \otimes_C B'$ morphism:

$$\begin{tikzcd}
 & A'\otimes_C B' & \\
A' \arrow[ur] & & B' \arrow[ul] \\
 & A\otimes_C B \arrow[uu, dashed, "\delta"] & \\
 & C \arrow[uul, bend left] \arrow[uur, bend right] & \\
A \arrow[uuu, dashed, "\alpha"] \arrow[uur, bend left] & & B \arrow[uuu, dashed, "\beta"'] \arrow[uul, bend right] \\
 & C \arrow[ul] \arrow[ur] \arrow[uu, dashed, "\gamma"] &
\end{tikzcd}$$

From now on in the paper, when A is an L-structure and α is an automorphism of A, then by (A,α) we mean the structure A expanded by the unary function corresponding to α, and A constantly denotes the L-structure.

Theorem 3.20. *Let $\mathscr{K}$ be a Fraïssé class of L-structures with canonical amalgamation. Then the class $\mathscr{H}$ of L-structures with automorphism is a Fraïssé class.*

Proof. $\mathscr{H}$ is obviously countable and has HP. It suffices to show that it has AP (JEP follows by taking C to be the empty structure). Take any $(A,\alpha),(B,\beta),(C,\gamma)\in\mathscr{H}$ such that (C,γ) embeds into (A,α) and (B,β). Then α,β,γ yield an automorphism η (as a natural transformation) of a cospan:

$$\begin{tikzcd}
A & & B \\
A \arrow[u, dashed, "\alpha"] & C \arrow[ul] \arrow[ur] & B \arrow[u, dashed, "\beta"'] \\
 & C \arrow[ul] \arrow[ur] \arrow[u, dashed, "\gamma"] &
\end{tikzcd}$$

Then, by the Fact 2.18, $\otimes_C(\eta)$ is an automorphism of the pushout diagram that looks exactly like the diagram in the second point of the Definition 3.19.

This means that the morphism $\delta : A\otimes_C B \to A\otimes_C B$ has to be automorphism. Thus, by the fact that the diagram commutes, we have the amalgamation of (A,α) and (B,β) over (C,γ) in $\mathscr{H}$. □

The following theorem is one of the most important in construction of the generic automorphism given in the next section. Together with canonical amalgamation it gives a general fact about Fraïssé classes, namely it says that expanding a Fraïssé class with an automorphism of the structures does not change the limit.

Theorem 3.21. *Let $\mathscr{C}$ be a Fraïssé class of finitely generated L-structures. Let $\mathscr{D}$ be the class of structures from $\mathscr{C}$ with additional unary function symbol* *interpreted as an automorphism of the structure. If $\mathscr{C}$ has the weak Hrushovski property and $\mathscr{D}$ is a Fraïssé class then the Fraïssé limit of $\mathscr{C}$ is isomorphic to the Fraïssé limit of $\mathscr{D}$ reduced to the language L.*

Proof. Let $\Gamma = \mathrm{Flim}(\mathscr{C})$ and $(\Pi, \sigma) = \mathrm{Flim}(\mathscr{D})$. By the Fraïssé Theorem 3.9 it suffices to show that the age of Π is $\mathscr{C}$ and that it is weakly ultrahomogeneous. The former comes easily, as for every structure $A \in \mathscr{C}$ we have the structure $(A, \mathrm{id}_A) \in \mathscr{D}$, which means that the structure A embeds into Π. On the other hand, if a structure $(B, \beta) \in \mathscr{D}$ embeds into (Π, σ), then obviously $B \in \mathscr{C}$ by the definition of $\mathscr{D}$. Hence, $\mathscr{C}$ is indeed the age of Π.

Now, to show that Π is weakly homogeneous, take any structures $A, B \in \mathscr{C}$ such that $A \subseteq B$ with a fixed embedding of A into Π. Without the loss of generality assume that $A = B \cap \Pi$ (i.e. A embeds into Π by inclusion). Let $\bar{A} \subseteq \Pi$ be the smallest substructure closed under the automorphism σ and containing A. It is finitely generated as an L-structure, as $\mathscr{C}$ is the age of Π. Let C be a finitely generated structure such that $\bar{A} \to C \leftarrow B$. Such structure exists by the JEP of $\mathscr{C}$. Again, we may assume without the loss of generality that $\bar{A} \subseteq C$. Then $\sigma \restriction_{\bar{A}}$ is a partial automorphism of C, hence by the WHP it can be extended to a structure $(\bar{C}, \gamma) \in \mathscr{D}$ such that $\gamma \restriction_{\bar{A}} = \sigma \restriction_{\bar{A}}$.

Then, by the weak ultrahomogeneity of (Π, σ) we can find an embedding g of $(\bar{C}, \gamma)$ such that the following diagram commutes:

$$\begin{tikzcd} (\bar{A}, \sigma \restriction_{\bar{A}}) \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & (\Pi, \sigma) \\ (\bar{C}, \gamma) \arrow[ur, "g"'] & \end{tikzcd}$$

Thus, we have that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & \bar{A} \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & \Pi \\ B \arrow[r, "f"] & C \arrow[r, "\subseteq"] & \bar{C} \arrow[u, "g"] \end{tikzcd}$$

which proves that Π is indeed a weakly ultrahomogeneous structure. Hence, it is isomorphic to Γ. □

Corollary 3.22. *Let $\mathscr{C}$ be a Fraïssé class of finitely generated L-structures with WHP and canonical amalgamation. Let $\mathscr{D}$ be the class consisting of structures from $\mathscr{C}$ with an additional automorphism. Let $\Gamma = \mathrm{Flim}(\mathscr{C})$ and $\Pi = \mathrm{Flim}(\mathscr{D})$. Then $\Gamma \cong \Pi|_L$.*

Proof. It follows from Theorems 3.20 and 3.21. □

4. Conjugacy classes in automorphism groups

Let M be a countable L-structure. Recall, we define a topology on the $G = \mathrm{Aut}(M)$: for any finite function $f : M \to M$ we have a basic open set $[f]_G = \{g \in G \mid f \subseteq g\}$.

4.1. **Prototype: pure set.** In this section, $M = (M, =)$ is an infinite countable set (with no structure beyond equality).

Remark 4.1. If $f_1, f_2 \in \mathrm{Aut}(M)$, then f_1 and f_2 are conjugate if and only if for each $n \in \mathbb{N} \cup \{\aleph_0\}$, f_1 and f_2 have the same number of orbits of size n.

Proof. It is easy to see. □

Theorem 4.2. *Let $\sigma \in \mathrm{Aut}(M)$ be an automorphism with no infinite orbit and with infinitely many orbits of size n for every $n > 0$. Then the conjugacy class of σ is comeagre in $\mathrm{Aut}(M)$.*

Proof. We will show that the conjugacy class of σ is an intersection of countably many comeagre sets.

Let $A_n = \{\alpha \in \mathrm{Aut}(M) \mid \alpha \text{ has infinitely many orbits of size } n\}$. This set is comeagre for every $n > 0$. Indeed, we can represent this set as an intersection of countable family of open dense sets. Let $B_{n,k}$ be the set of all finite functions $\beta \colon M \to M$ that consist of exactly k distinct n-cycles. Then:

$$\begin{aligned} A_n &= \{\alpha \in \mathrm{Aut}(M) \mid \alpha \text{ has infinitely many orbits of size } n\} \\ &= \bigcap_{k=1}^{\infty} \{\alpha \in \mathrm{Aut}(M) \mid \alpha \text{ has at least } k \text{ orbits of size } n\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{\beta \in B_{n,k}} [\beta]_{\mathrm{Aut}(M)}, \end{aligned}$$

where indeed, $\bigcup_{\beta \in B_{n,k}} [\beta]_{\mathrm{Aut}(M)}$ is dense in $\mathrm{Aut}(M)$: take any finite $\gamma \colon M \to M$ such that $[\gamma]_{\mathrm{Aut}(M)}$ is nonempty. Then also $\bigcup_{\beta \in B_{n,k}} [\beta]_{\mathrm{Aut}(M)} \cap [\gamma]_{\mathrm{Aut}(M)} \neq \emptyset$, one can easily construct a permutation that extends γ and have at least k many n-cycles.

Now we see that $A = \bigcap_{n=1}^{\infty} A_n$ is a comeagre set consisting of all functions that have infinitely many n-cycles for each n. The only thing left to show is that the set of functions with no infinite cycle is also comeagre. Indeed, for $m \in M$ let $B_m = \{\alpha \in \mathrm{Aut}(M) \mid m \text{ has finite orbit in } \alpha\}$. This is an open dense set. It is a union over basic open sets generated by finite permutations with m in their domain. Denseness is also easy to see.

Finally, by the Remark 4.1, we can say that

$$\sigma^{\mathrm{Aut}(M)} = \bigcap_{n=1}^{\infty} A_n \cap \bigcap_{m \in M} B_m,$$

which concludes the proof. □

4.2. **More general structures.**

Fact 4.3. *Suppose M is an arbitrary structure and $f_1, f_2 \in \mathrm{Aut}(M)$. Then f_1 and f_2 are conjugate if and only if $(M, f_1) \cong (M, f_2)$ as structures with one additional unary function that is an automorphism.*

Proof. Suppose that $f_1 = g^{-1} f_2 g$ for some $g \in \mathrm{Aut}(M)$. Then g is the isomorphism between (M, f_1) and (M, f_2). On the other hand if $g \colon (M, f_1) \to (M, f_2)$ is an isomorphism, then $g \circ f_1 = f_2 \circ g$ which exactly means that f_1, f_2 conjugate. □

Theorem 4.4. *Let $\mathscr{C}$ be a Fraïssé class of finitely generated L-structures. Let $\mathscr{D}$ be the class of structures from $\mathscr{C}$ with additional unary function symbol interpreted as an automorphism of the structure. If $\mathscr{C}$ has the weak Hrushovski property and $\mathscr{D}$ is a Fraïssé class, then there is a comeagre conjugacy class in the automorphism group of the* $\mathrm{Flim}(\mathscr{C})$*.*

Before we get to the proof, let us establish some notions. If $g\colon \Gamma \to \Gamma$ is a finite injective function, then we say that g is *good* if it gives (in a natural way) an isomorphism between $\langle \mathrm{Dom}(g)\rangle$ and $\langle \mathrm{Rng}(g)\rangle$, i.e. substructures generated by $\mathrm{Dom}(g)$ and $\mathrm{Rng}(g)$ respectively. Of course, in our case, g is good if and only if $[g]_{\mathrm{Aut}(\Gamma)} \neq \emptyset$ (because of ultrahomogeneity of Γ.

Also it is important to mention that an isomorphism between two finitely generated structures is uniquely given by a map from generators of one structure to the other. This allow us to treat a finite function as an isomorphism of finitely generated structure.

Proof. Let $\Gamma = \mathrm{Flim}(\mathscr{C})$ and $(\Pi, \sigma) = \mathrm{Flim}(\mathscr{D})$. First, by the Theorem 3.21, we may assume without the loss of generality that $\Pi = \Gamma$. Let $G = \mathrm{Aut}(\Gamma)$, i.e. G is the automorphism group of Γ. We will construct a strategy for the second player in the Banach-Mazur game on the topological space G. This strategy will give us a subset $A \subseteq G$ and as we will see a subset of the σ's conjugacy class. By the Banach-Mazur theorem (see 2.13) this will prove that this class is comeagre.

Recall, G has a basis consisting of open sets $\{g \in G \mid g \restriction_A = g_0 \restriction_A\}$ for some finite set $A \subseteq \Gamma$ and some automorphism $g_0 \in G$. In other words, a basic open set is a set of all extensions of some finite partial automorphism g_0 of Γ. By $B_g \subseteq G$ we denote a basic open subset given by a finite partial isomorphism g. Again, Note that B_g is nonempty because of ultrahomogeneity of Γ.

With the use of Corollary 2.17 we can consider only games where both players choose finite partial isomorphisms. Namely, player I picks functions $f_0, f_1, \ldots$ and player II chooses $g_0, g_1, \ldots$ such that $f_0 \subseteq g_0 \subseteq f_1 \subseteq g_1 \subseteq \ldots$, which identify the corresponding basic open subsets $B_{f_0} \supseteq B_{g_0} \supseteq \ldots$.

Our goal is to choose g_i in such a manner that $\bigcap_{i=0}^{\infty} B_{g_i} = \{g\}$ and (Γ, g) is ultrahomogeneous with age $\mathscr{D}$. By the Fraïssé theorem (see 3.9) it will follow that $(\Gamma, \sigma) \cong (\Gamma, g)$, thus by the Fact 4.3 we have that σ and g conjugate.

First, let us enumerate all pairs of structures $\{\langle (A_n, \alpha_n), (B_n, \beta_n)\rangle\}_{n\in\mathbb{N}}, \in \mathscr{D}$ such that the first element of the pair embeds by inclusion in the second, i.e. $(A_n, \alpha_n) \subseteq (B_n, \beta_n)$. Also, it may be that A_n is an empty. We enumerate the elements of the Fraïssé limit $\Gamma = \{v_0, v_1, \ldots\}$.

Fix a bijection $\gamma\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that for any $n, m \in \mathbb{N}$ we have $\gamma(n, m) \geq n$. This bijection naturally induces a well ordering on $\mathbb{N}\times\mathbb{N}$. This will prove useful later, as the main ingredient of the proof will be a bookkeeping argument.

For technical reasons, let $g_{-1} = \emptyset$ and $X_{-1} = \emptyset$. Suppose that player I in the n-th move chooses a finite partial automorphism f_n. We will construct a finite partial automorphism $g_n \supseteq f_n$ together with a finitely generated substructure $\Gamma_n \subseteq \Gamma$ and a set $X_n \subseteq \mathbb{N}^2$ such that the following properties hold:

(i) g_n is good and $\mathrm{Dom}(g_n) \cup \mathrm{Rng}(g_n) \subseteq \langle \mathrm{Dom}(g_n)\rangle = \Gamma_n$, i.e. g_n gives an automorphism of a finitely generated substructure Γ_n

(ii) $g_n(v_n)$ and $g_n^{-1}(v_n)$ are defined, Before we give the third point, suppose recursively that g_{n-1} already satisfy all those properties. Let us enumerate $\{\langle (A_{n,k},\alpha_{n,k}),(B_{n,k},\beta_{n,k}),f_{n,k}\rangle\}_{k\in\mathbb{N}}$ all pairs of finitely generated structures with automorphisms such that the first substructure embed into the second by inclusion, i.e. $(A_{n,k},\alpha_{n,k})\subseteq(B_{n,k},\beta_{n,k})$, and $f_{n,k}$ is an embedding of $(A_{n,k},\alpha_{n,k})$ in the (Γ_{n-1},g_{n-1}). We allow $A_{n,k}$ to be empty. Although g_{n-1} is a finite function, we may treat it as an automorphism as we have said before.

(iii) Let $(i,j)=\min\{(\{0,1,\dots\}\times\mathbb{N})\setminus X_{n-1}\}$ (with the order induced by γ). Then $X_n=X_{n-1}\cup\{(i,j)\}$ and $(B_{n,k},\beta_{n,k})$ embeds in (Γ_n,g_n) so that this diagram commutes:

$$\begin{tikzcd}
& (\Gamma_n,g_n) & \\
(B_{i,j},\beta_{i,j}) \arrow[ur, dashed, "\hat{f}_{i,j}"] & & (\Gamma_{n-1},g_{n-1}) \arrow[ul, dashed, "\subseteq"'] \\
& (A_{i,j},\alpha_{i,j}) \arrow[ul, "\subseteq"] \arrow[ur, "f_{i,j}"'] &
\end{tikzcd}$$

First, we will satisfy the item (iii). Namely, we will construct Γ_n',g_n' such that $g_{n-1}\subseteq g_n'$, $\Gamma_{n-1}\subseteq\Gamma_n'$, g_n' gives an automorphism of Γ_n' and $f_{i,j}$ extends to an embedding of $(B_{i,j},\beta_{i,j})$ to (Γ_n',g_n'). But this can be easily done by the fact, that $\mathscr{D}$ has the amalgamation property. Moreover, without the loss of generality we can assume that all embeddings are inclusions.

$$\begin{tikzcd}
& (\Gamma_n',g_n') & \\
(B_{i,j},\beta_{i,j}) \arrow[ur, dashed, "\subseteq"] & & (\Gamma_{n-1},g_{n-1}) \arrow[ul, dashed, "\subseteq"'] \\
& (A_{i,j},\alpha_{i,j}) \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

It is important to note that g_n' should be a finite function and once again, as it is an automorphism of a finitely generated structure, we may think it is simply a map from one generators of Γ_n' to the others. By the weak ultrahomogeneity of Γ, we may assume that $\Gamma_n'\subseteq\Gamma$.

Now, by the WHP of $\mathscr{K}$ we can extend $\langle\Gamma_n'\cup\{v_n\}\rangle$ together with its partial isomorphism g_n' to a finitely generated structure Γ_n together with its automorphism $g_n\supseteq g_n'$ and (again by weak ultrahomogeneity) without the loss of generality we may assume that $\Gamma_n\subseteq\Gamma$. This way we've constructed g_n that has all desired properties.

Now we need to see that $g=\bigcap_{n=0}^{\infty}g_n$ is indeed an automorphism of Γ such that (Γ,g) has the age $\mathscr{H}$ and is weakly ultrahomogeneous. It is of course an automorphism of Γ as it is defined for every $v\in\Gamma$ and is an union of an increasing chain of automorphisms of finitely generated substructures.

Take any $(B,\beta)\in\mathscr{D}$. Then, there are i,j such that $(B,\beta)=(B_{i,j},\beta_{i,j})$ and $A_{i,j}=\emptyset$. By the bookkeeping there was n such that $(i,j)=\min\{\{0,1,\dots n-$

$1\} \times \mathbb{N} \setminus X_{n-1}\}$. This means that (B,β) embeds into (Γ_n, g_n), hence it embeds into (Γ, g). Hence, the age of (Γ, g) is $\mathscr{H}$.

It is also weakly ultrahomogeneous. Having $(A,\alpha) \subseteq (B,\beta)$, and an embedding $f : (A,\alpha) \to (\Gamma, g)$, we may find $n \in \mathbb{N}$ such that $(i,j) = \min\{\{0,1,\dots n-1\} \times X_{n-1}\}$ and $(A,\alpha) = (A_{i,j}, \alpha_{i,j}), (B,\beta) = (B_{i,j}, \beta_{i,j})$ and $f = f_{i,j}$. This means that there is a compatible embedding of (B,β) into (Γ_n, g_n), which means we can also embed it into (Γ, g).

Hence, $(\Gamma, g) \cong (\Gamma, \sigma)$. By this we know that g and σ conjugate in G. As we stated in the beginning of the proof, the set A of possible outcomes of the game (i.e. possible g's we end up with) is comeagre in G, thus σ^G is also comeagre and σ is a generic automorphism, as it contains a comeagre set A. □

Theorem 4.5. *Let $\mathscr{C}$ be a Fraïssé class of finitely generated L-structures with WHP and canonical amalgamation. Then* $\mathrm{Flim}(\mathscr{C})$ *has a generic automorphism.*

Proof. It follows trivially from Corollary 3.22 and the above Theorem 4.4. □

4.3. **Properties of the generic automorphism.** Let $\mathscr{C}$ be a Fraïssé class of finitely generated L-structures with weak Hrushovski property and canonical amalgamation. Let $\mathscr{D}$ be the Fraïssé class (by the Theorem 4.5 of the structures of $\mathscr{C}$ with additional automorphism of the structure. Let $\Gamma = \mathrm{Flim}(\mathscr{C})$.

Proposition 4.6. *Let σ be the generic automorphism of Γ. Then the set of fixed points of σ is isomorphic to Γ.*

Proof. Let $S = \{x \in \Gamma \mid \sigma(x) = x\}$. First we need to show that it is an infinite. By the theorem 4.4 we know that (Γ, σ) is the Fraïssé limit of $\mathscr{D}$, thus we can embed finite L-structures of any size with identity as an automorphism of the structure into (Γ, σ). Thus S has to be infinite. Also, the same argument shows that the age of the structure is exactly $\mathscr{C}$. It is weakly ultrahomogeneous, also by the fact that (Γ, σ) is in $\mathscr{D}$. □

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