1. Introduction

2. Preliminaries

2.1. Descriptive set theory.

Definition 2.1. Suppose X is a topological space and $A \subseteq X$. We say that A is *meagre* in X if $A = \bigcup_{n\in\mathbb{N}} A_n$, where A_n are nowhere dense subsets of X (i.e. $\mathrm{Int}(\bar{A_n}) = \emptyset$).

Definition 2.2. We say that A is *comeagre* in X if it is a complement of a meagre set. Equivalently, a set is comeagre iff it contains a countable intersection of open dense sets.

Every countable set is nowhere dense in any T_1 space, so, for example, $\mathbb{Q}$ is meagre in $\mathbb{R}$ (though being dense), which means that the set of irrationals is comeagre. Another example is...

Definition 2.3. We say that a topological space X is a *Baire space* if every comeagre subset of X is dense in X (equivalently, every meagre set has empty interior).

Definition 2.4. Suppose X is a Baire space. We say that a property P *holds generically* for a point in $x \in X$ if $\{x \in X \mid P \text{ holds for } x\}$ is comeagre in X.

Definition 2.5. Let X be a nonempty topological space and let $A \subseteq X$. The *Banach-Mazur game of A*, denoted as $G^{\star\star}(A)$ is defined as follows: Players I and II take turns in playing nonempty open sets $U_0, V_0, U_1, V_1, \ldots$ such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. We say that player II wins the game if $\bigcap_n V_n \subseteq A$.

There is an important theorem on the Banach-Mazur game: A is comeagre iff II can always choose sets $V_0, V_1, \ldots$ such that it wins. Before we prove it we need to define notions necessary to formalise and prove the theorem.

Definition 2.6. T is *the tree of all legal positions* in the Banach-Mazur game $G^{\star\star}(A)$ when T consists of all finite sequences $(W_0, W_1, \ldots, W_n)$, where W_i are nonempty open sets such that $W_0 \supseteq W_1 \supseteq \ldots \supseteq W_n$. In another words, T is a pruned tree on $\{W \subseteq X \mid W \text{is open nonempty}\}$.

By $[T]$ we denote the set of all "infinite branches" of T, i.e. infinite sequences $(U_0, V_0, \ldots)$ such that $(U_0, V_0, \ldots U_n, V_n) \in T$ for any $n \in \mathbb{N}$.

Definition 2.7. A *strategy* for II in $G^{\star\star}(A)$ is a subtree $\sigma \subseteq T$ such that

(i) σ is nonempty,
(ii) if $(U_0, V_0, \ldots, V_n) \in \sigma$, then for all open nonempty $U_{n+1} \subseteq V_n$, $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma$,
(iii) if $(U_0, V_0, \ldots, U_n) \in \sigma$, then for unique V_n, $(U_0, V_0, \ldots, U_n, V_n) \in \sigma$.

Intuitively, a strategy σ works as follows: I starts playing U_0 as any open subset of X, then II plays unique (by (iii)) V_0 such that $(U_0, V_0) \in \sigma$. Then I responds by playing any $U_1 \subseteq V_0$ and II plays uniqe V_1 such that $(U_0, V_0, U_1, V_1) \in \sigma$, etc.

Definition 2.8. A strategy σ is a *winning strategy for II* if for any game $(U_0, V_0 \ldots) \in [\sigma]$ (where $[\sigma]$ is defined analogically to $[T]$) player II wins, i.e. $\bigcap_n V_n \subseteq A$.

Now we can state the key theorem.

Theorem 2.9. *Let X be a nonempty topological space and let $A \subseteq X$. Then A is comeagre $\iff$ II has a winning strategy in $G^{\star\star}(A)$.*

In order to prove it we add an auxilary definition and lemma.

Definition 2.10. Let S be a pruned subtree of a strategy σ and let $p = (U_0, V_0, \dots, V_n) \in S$. We say that S is *comprehensive for p* if the family $\mathscr{V}_p = \{V_{n+1} \mid (U_0, V_0, \dots, V_n, U_{n+1}, V_{n+1}) \in S\}$ (it may be that $n = -1$) is pairwise disjoint and $\bigcup \mathscr{V}_p$ is dense in V_n.

We say that S is *comprehensive* if it is comprehensive for any $p = (U_0, V_0, \dots, V_n) \in S$.

Lemma 2.11. *Let S be a comprehensive pruned subtree of a strategy σ. Then:*

(i) For any V_n such that there is $p = (U_0, V_0, \dots, V_n) \in S$, this p is unique.
(ii) Let $W_n = \{V_n \mid (U_0, V_0, \dots, V_n) \in S\}$, i.e. W_n is a family of all possible choices player II can make in its n-th move. Then $\bigcup W_n$ is open and dense in X.
(iii) There exists such comprehensive $S \subseteq \sigma$.

Proof. (i): Suppose that there are some $p = (U_0, V_0, \dots, U_n, V_n)$, $p' = (U'_0, V'_0, \dots, U'_n, V'_n)$ such that $V_n = V'_n$ and $p \neq p'$. Let k be the smallest index such that those sequences differ. We have two possibilities:

- $U_k = U'_k$ and $V_k \neq V'_k$ – this cannot be true simply by the fact that S is a subset of a strategy.
- $U_k \neq U'_k$: by the comprehensiveness of S we know that for $q = (U_0, V_0, \dots, U_{k-1}, V_{k-1})$ the set $\mathscr{V}_q$ is pairwise disjoint. Thus $V_k \cap V'_k = \emptyset$, because $V_k, V'_k \in \mathscr{V}_q$. But this leads to a contradiction – V_n cannot be a nonempty subset of both V_k, V'_k.

(ii): The lemma is proved by induction on n. For $n = 0$ it follows trivially from the definition of comprehensiveness. Now suppose the lemma is true for n. Then the set $\bigcup_{V_n \in W_n} \bigcup \mathscr{V}_{p_{V_n}}$ (p_{V_n} is given uniquely from (i)) is dense and open in X by the induction hypothesis. But $\bigcup W_{n+1}$ is its superset, thus $\bigcup W_{n+1}$ is dense and open in X.

(iii): We construct S recursively as follows:

(1) $\emptyset \in S$,
(2) if $(U_0, V_0, \dots, U_n) \in S$, then $(U_0, V_0, \dots, U_n, V_n) \in S$ for the unique V_n given by the strategy σ,
(3) let $p = (U_0, V_0, \dots, V_n) \in S$, let $U^\star_{n+1}$ be the unique set player II whould play by σ given that player I played $U_{n+1} \subseteq V_n$. Now, by Zorn's Lemma, let $\mathscr{U}_p$ be a maximal collection of nonempty open subsets $U_{n+1} \subseteq V_n$ such that the set $\{U^\star_{n+1} \mid U_{n+1} \in \mathscr{U}_p\}$ is pairwise disjoint. Then put in S all $(U_0, V_0, \dots, V_n, U_{n+1}, U^\star_{n+1})$ such that $U_{n+1} \in \mathscr{U}_p$. This way S is comprehensive for p: the family $\mathscr{V}_p = \{V_{n+1} \mid (U_0, V_0, \dots, V_n, U_{n+1}, V_{n+1})\, S\}$ is exactly $\{U^\star_{n+1} \mid U_{n+1} \in \mathscr{U}_p\}$, which is pairwise disjoint and $\bigcup \mathscr{V}_p$ is obviously dense in V_n by it's maximality – if there was any open set $\tilde{U}_{n+1} \subseteq V_n$ disjoint from $\bigcup \mathscr{U}_p$, then the family $\mathscr{U}_p \cup \{\tilde{U}_{n+1}\}$ violates the maximality of $\mathscr{U}_p$.

$\square$

Now we can move to the proof of the Banach-Mazur theorem.

Proof. $\Rightarrow$: Let (A_n) be a sequence of dense open sets with $\bigcap_n A_n \subseteq A$. The simply II plays $V_n = U_n \cap A_n$, which is nonempty by the denseness of A_n.

$\Leftarrow$: Suppose II has a winning strategy σ. We will show that A is comeagre. Suppose we have a comprehensive $S \subseteq \sigma$. We claim that $\mathscr{W} = \bigcap_n \bigcup W_n \subseteq A$. By 2.11, (ii) sets $\bigcup W_n$ are open and dense, thus A must be comeagre. Now we prove the claim.

(A.a.) Suppose there is $x \in \mathscr{W}$ that is not in A. We will prove by induction that for any n there is exactly one $V_n \in W_n$ such that $x \in V_n$. For $n = 0$ this follows trivially by the comprehensiveness of S. Now suppose that there is exactly one $V_n \in W_n$ such that $x \in V_n$. By our assumption there is a $V'_{n+1} \in W_{n+1}$ such that $x \in V'_{n+1}$. By 2.11 we have unique $p_{V'_{n+1}} = (U'_0, V'_0, \ldots, V'_{n+1}) \in S$. It must be that $x \in V'_n$, so by the induction hypothesis $V'_n = V_n$, thus $V'_{n+1} \in \mathscr{V}_{p_{V_n}}$. But the family $\mathscr{V}_{p_{V_n}}$ is disjoint, hence $V_{n+1} = V'_{n+1}$ is unique.

Now the game $(U_0, V_0, U_1, V_1, \ldots) = \bigcup_n p_{V_n} \in [S] \subseteq [\sigma]$ where $x \in V_0, V_1, \ldots$ is not winning for player II, which contradicts the assumption that σ is a winning strategy. □

Pytania:

- Czy da się coś zrobić, żeby $\mathscr{V}$ nie było takie brzydkie?
- Jak to napisać, że się zrzyna z książki?
- Dodatkowy przykład pod def 2.2
- $G^{\star\star}(A)$ czy $G^{**}(A)$? Czy może $G^{**}(X,A)$? Jakiś skrót na to?
- w 2.11 (i), jak to ładniej sformułować?
- w 2.11 (iii), może to wyodrębnić? Może to dać jako pierwsze, a pierwsze dwa później?
- dodać tytuł do 2.9
- czy w dowodzie twierdzenia napisać jeszcze raz co to jest W_n?

2.2. Fraïssé classes.

Fact 2.12 (Fraïssé theorem). *Then there exists a unique up to isomorphism counable L-structure M such that...*

Definition 2.13. For $\mathscr{C}$, M as in Fact 2.12, we write $\mathrm{FLim}(\mathscr{C}) := M$.

Fact 2.14. *If $\mathscr{C}$ is a uniformly locally finite Fraïssé class, then* $\mathrm{FLim}(\mathscr{C})$ *is $\aleph_0$-categorical and has quantifier elimination.*

3. Conjugacy classes in automorphism groups

3.1. Prototype: pure set.

In this section, $M = (M, =)$ is an infinite countable set (with no structure beyond equality).

Proposition 3.1. *If $f_1, f_2 \in \mathrm{Aut}(M)$, then f_1 and f_2 are conjugate if and only if for each $n \in \mathbb{N} \cup \{\aleph_0\}$, f_1 and f_2 have the same number of orbits of size n.*

Proposition 3.2. *The conjugacy class of $f \in \mathrm{Aut}(M)$ is dense if and only if...*

Proposition 3.3. *If $f \in \mathrm{Aut}(M)$ has an infinite orbit, then the conjugacy class of f is meagre.*

Proposition 3.4. *An automorphism f of M is generic if and only if...*

Proof. □

3.2. **More general structures.**

Proposition 3.5. *Suppose M is an arbitrary structure and $f_1, f_2 \in \mathrm{Aut}(M)$. Then f_1 and f_2 are conjugate if and only if $(M, f_1) \cong (M, f_2)$.*

Definition 3.6. We say that a Fraïssé class $\mathscr{C}$ has *weak Hrushovski property* (*WHP*) if for every $A \in \mathscr{C}$ and partial automorphism $p : A \to A$, there is some $B \in \mathscr{C}$ such that p can be extended to an automorphism of B, i.e. there is an embedding $i : A \to B$ and a $\bar{p} \in \mathrm{Aut}(B)$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\bar{p}"] & B \\ A \arrow[u, "i"] \arrow[r, "p"] & A \arrow[u, "i"'] \end{tikzcd}$$

Proposition 3.7. *Suppose $\mathscr{C}$ is a Fraïssé class in a relational language with WHP. Then generically, for an $f \in \mathrm{Aut}(\mathrm{FLim}(\mathscr{C}))$, all orbits of f are finite.*

Proposition 3.8. *Suppose $\mathscr{C}$ is a Fraïssé class in an arbitrary countable language with WHP. Then generically, for an $f \in \mathrm{Aut}(\mathrm{FLim}(\mathscr{C}))$ ...*

3.3. **Random graph.**

Definition 3.9. The *random graph* is...

Fact 3.10. *The*

Proposition 3.11. *Generically, the set of fixed points of $f \in \mathrm{Aut}(M)$ is isomorphic to M (as a graph).*